NCERT SOLUTIONS
CLASS-XII MATHS
CHAPTER-6
APPLICATIONS OF DERIVATIVE

Excercise 1

Q-1) For a circle, with radius given below, find the rate of change of the area w.r.t raduis'a'.
i) a = 2mm,

ii) a = 5mm

Ans.) We know that the area of circle with radius 'a' is πa^2.

i.e. A = πa^2

The rate of change of area w.r.t 'a' is,

$\frac{dA}{da} = \frac{d(\pi a^2)}{da} = 2\pi a$

a = 2mm

$\frac{dA}{da} = 2\pi(2) = 4\pi$

Thus, $4\pi(mm^2/s)$ is the rate of change of area of circle, when a = 2mm.

a = 5mm

$\frac{dA}{da} = 2\pi(5) = 10\pi$

Thus, $10\pi(mm^2/s)$ is the rate of change of area of circle, when a = 5mm.

Q-2) Find the increase in surface area of a cube, given that cube's volume is increasing at the rate of $27(mm^2/s)$ and the length of the edge is 18mm.

Ans.) Lets assume the length of edge is y*mm,* surface area as S and volume as V.

So, V = y^3, S = y^2 ; y is a function of time x.

In the question its is given that $\frac{dV}{dx} = 27(mm^3/s)$.

Applying chain rule, we get

$27 = \frac{dV}{dx} = \frac{d(y^3)}{dx} = \frac{d(y^3)}{dy} * \frac{dy}{dx} = 3y^2 * \frac{dy}{dx}$

$\frac{dy}{dx} = 27/3y^2$ .... (a)

Applying chain rule on surface area, we get

$\frac{dS}{dx} = \frac{d(6y^2)}{dx} = \frac{d(6y^2)}{dy} * \frac{dy}{dx} = 12y * \frac{dy}{dx}$

$= 12y(27/3y^2) = 108/y\ (because\ from\ (a))$

Here its is given that y = 18mm

$\frac{dS}{dx} = 108/18 = 6mm^2/s$

Thus, the surface area of a circle is increasing at the rate of $6mm^2/s$ when the edge length is 18mm

Q-3) For a circle, determine the rate of change of area when radius 'y' is 20mm. The radius of circle is increase at the rate of 6mm/s.

Ans.) We know that the area of circle with radius 'y' is πy^2.

i.e. A = πy^2

The rate of change of area w.r.t time 'x' is,

Applying chain rule in above equation, we get

$\frac{dA}{dx} = \frac{d(\pi y^2)}{dx} = \frac{d(\pi y^2)}{dy} * \frac{dy}{dx} = 2\pi y\frac{dy}{dx}$

Here, given that, $\frac{dy}{dx} = 6mm/s$

$\frac{dA}{dx} = 2\pi y(6) = 12\pi y$

Also given that radius y = 20mm

$\frac{dA}{dx} = 12\pi(20) = 240mm^2/s$

Thus, $240mm^2/s$ is the rate of change of area of circle, when y = 20mm.

Q-4) For a variable cube, find the increase in volume of a cube when the length of the edge is 20mm, given that cube's edge is increasing at the rate of 6mm/s.

Ans.) Lets assume the length of edge is y*mm,* and volume as V.

So, V = y^3 ; y is function of time 'x'.

Then, By applying chain rule we get,

$\frac{dV}{dx} = 3y^2 * \frac{dy}{dx}$

Here it is given that,

$\frac{dy}{dx} = 6mm/s \ \frac{dV}{dx} = 3y^2(6) = 18y^2$

Also given that edge length is y = 20mm

$\frac{dV}{dx} = 18(20^2) = 7200mm^3/s$

Thus, the Volume of a variable cube is increasing at the rate of $7200mm^3/s$ when the edge length is 20mm.

Q-5) When a stone is thrown into a steady pond the circular waves are created and they started travelling at the rate of 10mm/s. Find how fast the area enclosed by circular wave is increasing at an instant when the radius of the circle is 16mm.

Ans.) We know that the area of circle with radius 'y' is πy^2.

i.e. A = πy^2

The rate of change of area w.r.t time 'x' is,

Applying chain rule in above equation, we get

$\frac{dA}{dx} = \frac{d(\pi y^2)}{dx} = \frac{d(\pi y^2)}{dy} * \frac{dy}{dx} = 2\pi y \frac{dy}{dx}$

Here, given that, $\frac{dy}{dx} = 10mm/s$

$\frac{dA}{dx} = 2\pi y(10) = 20\pi y$

Also given that radius y = 16mm

$\frac{dA}{dx} = 20\pi(16) = 320mm^2/s$

Thus, $320mm^2/s$ is the rate of change of area enclosed by circular wave, when y = 16mm.

Q-6) For a circle, find the rate at which the circumference is increasing given that its radius is increasing at the rate of 1.4mm/s.

Ans.) We know that the area of circle with radius 'y' is

C = $2\pi y$

The rate of change of circumference w.r.t time 'x' is obtained by applying chain rule,

$\frac{dC}{dx} = \frac{d(2\pi y)}{dx} = \frac{d(2\pi y)}{dy} * \frac{dy}{dx} = 2\pi \frac{dy}{dx}$

Here its is given that, $\frac{dy}{dx} = 1.4mm/s$

$\frac{dC}{dx} = 2\pi(1.4) = 2.8mm/s$

Thus, 2.8mm/s is the rate of increase in circumference.

Q-7) Find the rate at which the perimeter and area of rectangle is changing, when a = 16mm, b = 12mm. Given that the length 'a' of a rectangle is increasing at 10mm/min and width of a rectangle is decreasing at 8mm/min.

Ans.) Here given that rectangle is increasing at 10mm/min and width of a rectangle is decreasing at 8mm/min,

$\frac{da}{dx} = 10mm/min \ and \ \frac{db}{dx} = -8mm/min$

Perimeter of rectangle, P = 2(a + b)

The rate of change of perimeter w.r.t time 'x' is,

$\frac{dP}{dx} = 2(\frac{da}{dx} + \frac{db}{dx}) = 2(10-8) = 4mm/min$

Thus, the perimeter of a rectangle is increasing at 4mm/min.

Area of rectangle, A = a*b

The rate of change of area w.r.t time 'x' is,

$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{\mathrm{d}a}{\mathrm{d}x} * b + \frac{\mathrm{d}a}{\mathrm{d}x} * a = 10b - 8a$

Here, when a = 16mm and b = 12mm,

$\frac{\mathrm{d}A}{\mathrm{d}x} = 10(12) - 8(16) = 120 - 128 = -8mm^2/min$

Thus, the area of rectangle is decreasing at $8mm^2/min$.

Q-8) A football is being inflated by pumping 1600cc of air per second. Find the rate at which the radius of football will increase at an instant when radius y = 10cm.

Ans.) As we know the shape of a football is spherical.

The volume(V) of sphere is given by,

$V = \frac{4}{3}\pi y^3$

By applying chain rule in above equation by differentiating w.r.t time 'x', we get

$\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{\mathrm{d}(\frac{4}{3}\pi y^3)}{\mathrm{d}x} = \frac{\mathrm{d}(\frac{4}{3}\pi y^3)}{\mathrm{d}y} * \frac{\mathrm{d}y}{\mathrm{d}x} = 4\pi y^2 \frac{\mathrm{d}y}{\mathrm{d}x}$

Here, $\frac{\mathrm{d}V}{\mathrm{d}x} = 1600cc/s$ (given)

$1600 = 4\pi y^2 \frac{\mathrm{d}y}{\mathrm{d}x}\ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1600}{4\pi y^2} = \frac{400}{\pi y^2}$

Here radius y = 10cm (given)

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{400}{\pi 10^2} = \frac{4}{\pi} cm/s$

Thus, radius of football is increasing at $\frac{4}{\pi} cm/s$ when radius y = 10cm .

Q-9) Find out the rate of change of volume of a football w.r.t its radius at an instant when its radius is 200mm. The radius of the spherically shaped football is variable.

Ans.) As we know the shape of a football is spherical.

The volume(V) of sphere is given by,

$V = \frac{4}{3}\pi x^3$

By applying chain rule in above equation by differentiating w.r.t radius 'x', we get

$\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{\mathrm{d}(\frac{4}{3}\pi x^3)}{\mathrm{d}x} = 4\pi x^2$

Here, radius is given 200mm,

$\frac{\mathrm{d}V}{\mathrm{d}x} = 4\pi(200)^2 = 160000\pi\ mm^3/s$

Thus the volume of a football is increasing at the rate of $160000\pi\ mm^3/s$.

Q-10) An object travels along the curve 12a = $12b = 2a^3 + 4$.Determine the points on the curve where the 'a' co-ordinate is (1/8)th of the 'b' co-ordinate.

Ans.) Here it is given that the equation of the curve is,

$12b = 2a^3 + 4$

By differentiating above equation w.r.t. time 'x, we get

$12\frac{\mathrm{d}b}{\mathrm{d}x} = 6a^2\frac{\mathrm{d}a}{\mathrm{d}x} + 0$

$2\frac{\mathrm{d}b}{\mathrm{d}x} = a^2\frac{\mathrm{d}a}{\mathrm{d}x}$...............................(1)

Here it is given that,'a' co-ordinate is (1/8)th of the 'b' co-ordinate

i.e. $\frac{1}{8}\frac{\mathrm{d}b}{\mathrm{d}x} = \frac{\mathrm{d}a}{\mathrm{d}x}$

$\frac{\mathrm{d}b}{\mathrm{d}x} = 8\frac{\mathrm{d}a}{\mathrm{d}x}$

Putting this value in equation (1)

$2(8\frac{\mathrm{d}a}{\mathrm{d}x}) = a^2\frac{\mathrm{d}a}{\mathrm{d}x}\ 16\frac{\mathrm{d}a}{\mathrm{d}x} = a^2\frac{\mathrm{d}a}{\mathrm{d}x}\ (a^2 - 16)\frac{\mathrm{d}a}{\mathrm{d}x} = 0\ a^2 = 16\ a = \pm 4$

Now, for a = – 4

$b = \frac{2(-4)^3+4}{12} = \frac{132}{12} = 11$

For, a = 4

$b = \frac{2(4)^3+4}{12} = \frac{124}{12} = \frac{-31}{3}$

Thus, the required co-ordinate or points are $\left(-4, \frac{-31}{3}\right)\ and\,(4, 11)$.

Q-11) A wall has a stick of 13m leaning on it. The base of the stick is pulled away from wall along the ground at 4cm/s. Find out at which rate the height of the stick will decrease when the bottom point of the stick is 5m far from the wall.

Ans.) Let a *m* be the distance from bottom of stick to wall and b *m* be the height of the wall where the stick is in contact with the wall.

Thus, we can apply Pythagoras theorem in this case, then we get

$a^2 + b^2 = 169$

(stick is of 13m length)

$b = \sqrt{169 - a^2}$

By differentiating above equation w.r.t time 'x' we will get rate of change of height,

$\frac{\mathrm{d}b}{\mathrm{d}x} = \frac{-a}{\sqrt{169-a^2}} * \frac{\mathrm{d}a}{\mathrm{d}x}$

Here we have $\frac{\mathrm{d}a}{\mathrm{d}x} = 4cm/s$

$\frac{\mathrm{d}b}{\mathrm{d}x} = \frac{-4a}{\sqrt{169-a^2}}$

Here, the bottom point of the stick is 5m far from the wall, a = 5m

$\frac{\mathrm{d}b}{\mathrm{d}x} = \frac{-4(5)}{\sqrt{169-(5)^2}} = \frac{-20}{12} = \frac{-5}{3}$

Thus, the height of the stick is decreasing at the rate of (-5/3)cm/s.

Q-12) Find the rate at which the volume of a balloon is increasing at an instant when its radius is 2cm. The radius of balloon is increasing at the rate of 1cm/s.

Ans.) As we know the shape of a balloon is spherical.

The volume(V) of sphere with radius 'y' is given by,

$V = \frac{4}{3}\pi y^3$

By applying chain rule in above equation by differentiating w.r.t time 'x', we get

$\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{\mathrm{d}(\frac{4}{3}\pi y^3)}{\mathrm{d}x} = \frac{\mathrm{d}(\frac{4}{3}\pi y^3)}{\mathrm{d}y} * \frac{\mathrm{d}y}{\mathrm{d}x} = 4\pi y^2\frac{\mathrm{d}y}{\mathrm{d}x}$

Here, given that,

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1cm/s\ \frac{\mathrm{d}V}{\mathrm{d}x} = 4\pi y^2(1) = 4\pi y^2 cm^3/s$

Also given that the radius y = 2cm,

$\frac{\mathrm{d}V}{\mathrm{d}x} = 4\pi(2)^2 = 8\pi cm^3/s$

Thus, the volume of balloon is increasing at the rate of $8\pi cm^3/s$.

Q-13) A football is having radius $\frac{5}{2}(4a+1)$ which is variable. Find the rate of change in volume of football w.r.t. 'a'.

Ans.) As we know the shape of a football is spherical.

The volume(V) of sphere with radius 'y' is given by,

$V = \frac{4}{3}\pi y^3$

Here it is given that y = $\frac{5}{2}(4a+1)$

$$V = \frac{4}{3}\pi(\frac{5}{2}(4a+1))^3 = \frac{125}{6}\pi(4a+1)^3$$

Differentiating above equation w.r.t 'a', we get,

$$\frac{dV}{da} = \frac{d\frac{125}{6}\pi(4a+1)^3}{da} = \frac{125}{6}\pi * 3(4a+1)^2 * 4 = 250(4a+1)^2$$

Q-14) A wheat flour is being pouring from a sprue at $24cm^2/s$. The fallen wheat flour forms a conical shape in the ground in such a way that the radius of the cone at base is 6 times the height of the cone. Then find out the rate of increase in the height of cone at an instant when the height is 8cm.

Ans.) Let V be the volume of cone , x be the radius of base of cone and y be the height of the cone which is increasing.

So, volume of cone is given by,

$$V = \frac{1}{3}\pi x^2 y$$

Here , x = 6y is given.

$$V = \frac{1}{3}\pi(6y)^2 y = 12\pi y^3$$

Now, the rate of increase in the height of cone is obtained by differentiating above equation w.r.t. time 't'

$$\frac{dV}{dt} = 12\pi\frac{dy^3}{dy} * \frac{dy}{dt} = 36\pi y^2\frac{dy}{dt}$$

Here, $\frac{dV}{dt} = 24cm^3/s$

$$24 = 36\pi y^2\frac{dy}{dt}$$

Also the y = 8cm is given,

$$24 = 36\pi(8)^2\frac{dy}{dt}\ \frac{dy}{dt} = \frac{1}{96\pi}$$

Thus, the height of wheat flour cone is increasing at the rate of $\frac{1}{96\pi}cm/s$.

Q-15) S(x) = $0.014a^3 – 0.006a^2 + 30a + 8000$

The above equation represents the total cost of the 'a' units manufactured.

Give the value of the marginal cost when 34 units are manufactured.

Ans.) Marginal cost(MC) is defined as the rate of change of total cost w.r.t. the numbers of units manufactured.

MC = $\frac{dC}{da} = \frac{d(0.014a^3 – 0.006a^2 + 30a + 8000)}{da} = 0.042a^2 – 0.012a + 30$

Here, a = 34 is given.

$$MC = \frac{dC}{da} = 0.042a^2 – 0.012a + 30 = 0.042(34)^2 – 0.012(34) + 30 = 78.144$$

Thus, marginal cost for 34 units manufactured is Rs.78.144.

Q-16) C(x) = $36y^2 + 52y + 30$

The above equation shows the revenue generated from y units sold.

Then give the value of the marginal revenue for 14 units sold.

Ans.) Marginal Revenue(MR) is defined as the rate of change of total revenue generated w.r.t. numbers of units sold.

$$MR = \frac{dC}{dy} = \frac{d(36y^2 + 52y + 30)}{dy} = 72y + 52$$

Here, y = 14 units (given)

MR = 72(14) + 52 = 1060

Thus, marginal revenue for 14 units sold is Rs.1060.

Q-17) For a circle the values for rate of change of area w.r.t. its radius at an instant when its radius is 12cm is given, find the correct option from the options given below.

(i) 20π

(ii) 10π

(iii) 40π

(iv) 24π

Ans.) We know that the area of circle with radius 'a' is πa^2.

i.e. A = πa^2

The rate of change of area w.r.t 'a' is,

$\frac{\mathrm{d}A}{\mathrm{d}a} = \frac{\mathrm{d}(\pi a^2)}{\mathrm{d}a} = 2\pi a$

Here a = 12cm is given,

$\frac{\mathrm{d}A}{\mathrm{d}a} = 2\pi(12) = 24cm^2/s$

Thus, Option (iv) is correct.

Q-18) C(x) = $6y^2 + 72y + 10$

The above equation shows the revenue generated from y units sold.

Then give the value of the marginal revenue for 30 units sold from the options given below.

(i) 118

(ii) 126

(iii) 432

(iv) 180

Ans.) Marginal Revenue(MR) is defined as the rate of change of total revenue generated w.r.t. numbers of units sold.

$MR = \frac{\mathrm{d}C}{\mathrm{d}y} = \frac{\mathrm{d}(6y^2+72y+10)}{\mathrm{d}y} = 12y + 72$

Here, y = 30 units (given)

MR = 12(30) + 72 = 432

Thus, marginal revenue for 30 units sold is Rs.432

Thus, option (iii) is correct.

Excercise 2 :

Q-1) Show that the function given below is strictly increasing on R.

g(y) = 4y + 19

Ans.)

Assuming y_1 and y_2 are the two real numbers.

Thus,

$y_1 < y_2$à $4y_1 < 4y_2$à $4y_1 + 19 < 4y_2 + 19$ à $g(y_1) < g(y_2)$

Thus, the given function g is strictly increasing on R.

This problem can be solved by Alternate method as follows,

g'(y) = 4 > 0, for every interval of R.

Thus, the given function g is strictly increasing on R.

Q-2) Show that the function given below is strictly increasing on R.

g(y) = e^{3y}

Ans.)

Assuming y_1 and y_2 are the two real numbers.

Thus,

$y_1 < y_2$à $3y_1 < 3y_2$à $e^{3y_1} < e^{3y_2}$à $g(y_1) < g(y_2)$

Thus, the given function g is strictly increasing on R.

Q-3) Show that the function given below is strictly increasing on R.

g(y) = $\sin y$

(i) g(y) is strictly decreasing in $(\frac{\pi}{2}, \pi)$

(ii) g(y) is strictly increasing in $(0, \frac{\pi}{2})$

(iii) g(y) is neither decreasing nor increasing in $(0, \pi)$

Ans.)

Here, g(y) = $\sin y$

Now, g'(y) = $\cos y$

(i) As we know that for each $y \epsilon (\frac{\pi}{2}, \pi),\ \cos y < 0$, we get g'(y) < 0.

Thus, g(y) is strictly decreasing in $(\frac{\pi}{2}, \pi)$.

(ii) As we know that for each $y \epsilon (0, \frac{\pi}{2}),\ \cos y > 0$, we get g'(y) > 0.

Thus, g(y) is strictly increasing in $(\frac{\pi}{2}, \pi)$.

(iii) From the (i) and (ii) it is clear that that g(y) is neither decreasing nor increasing in $(0, \pi)$.

Q-4) Find out in which intervals the given function is

(i) Strictly decreasing

(ii) Strictly increasing

g(y) = 3y^2 – 5y

Ans.)

Here, g(y) = $3y^2 - 5y$

g'(y) = 6y – 5

Now, g'(y) = 0 à y = 5/6

Here, the point 5/6 splits the real line into 2 disjoint interval and they are $(-\infty, \frac{5}{6})\ and\ (\frac{5}{6}, \infty)$.

For interval $(-\infty, \frac{5}{6})$,

g'(y) = 6y – 5 < 0

Thus, g(y) is strictly decreasing in $(-\infty, \frac{5}{6})$.

For interval $(\frac{5}{6}, \infty)$,

g'(y) = 6y – 5 > 0

Thus, g(y) is strictly increasing in $(-\infty, \frac{5}{6})$.

Q-5) Find out in which intervals the given function is

(i) Strictly decreasing

(ii) Strictly increasing

g(y) = 2y^3 – 3y^2 – 36y + 7

Ans.)

Here, g(y) = $2y^3 - 3y^2 - 36y + 7$

g'(y) = $6y^2 - 6y - 36 = 6(y^2 - y - 6) = 6(y + 3)(y + 2)$

Thus, g'(y) = 0,

y = -2,3

Thus, these two points -2 and 3 divides the real line in 3 disjoint intervals and they are $(-\infty, -2), (-2, 3)\ and\ (3, \infty)$.

For interval $(-\infty, -2)$

g'(y) = 6(y + 3)(y + 2) > 0

Thus, g(y) is strictly increasing in $(-\infty, -2)$

For interval (- 2,3)

g'(y) = 6(y + 3)(y + 2) < 0

Thus, g(y) is strictly decreasing in (- 2,3)

For interval $(3, \infty)$

g'(y) = 6(y + 3)(y + 2) > 0

Thus, g(y) is strictly increasing in $(3, \infty)$.

Q-6)Find out the intervals in which the functions given below are strictly increasing on R.

(i) 9 – 5y – y^2

(ii) y^2 + 4y – 7

(iii) $(y + 5)^3(y - 7)^3$

(iv) -2y^3 – 9y^2 – 12y + 1

(v) 7 – 7y – y^2

Ans.)

(i) Here, g(y) = 9 – 5y –y^2

g'(y) = – 5 – 2y

Now, g'(y) = 0

y = -5/2

Here, the point -5/2 splits the real line into 2 disjoint interval and they are $(-\infty, \frac{-5}{2})\ and\ (\frac{-5}{2}, \infty)$

For interval $(-\infty, \frac{-5}{2})$,

g'(y) = – 5 – 2y > 0

Thus, g(y) is strictly increasing in $(-\infty, \frac{-5}{2})$.

For interval $(\frac{-5}{2}, \infty)$,

g'(y) = – 5 – 2y< 0

Thus, g(y) is strictly decreasing in $(-\infty, \frac{-5}{2})$.

(ii) Here, g(y) = y^2 + 4y – 7

g'(y) = 2y + 4

Now, g'(y) = 0

y = – 2

Thus, this point divides the real line in 2 disjoint intervals and they are $(-\infty, -2),\ and\ (-2, \infty)$.

For interval $(-\infty, -2)$

g'(y) = 2y + 4 < 0

Thus, g(y) is strictly decreasing in $(-\infty, -2)$

For interval $(-2, \infty)$

g'(y) = 2y + 4 > 0

Thus, g(y) is strictly increasing in $(-2, \infty)$

(iii) Here, g(y) =$(y + 5)^3(y - 7)^3$

g'(y) = $3(y + 5)^2(y - 7)^3 + 3(y + 5)^3(y - 7)^2$

= $3(y + 5)^2(y - 7)^2(y + 5 + y - 7)$

= $3(y + 5)^2(y - 7)^2(2y - 2)$

= $6(y + 5)^2(y - 7)^2(y - 1)$

Now, g'(y) = 0

y = -5,1 and 7

Thus, these three points -5, 1 and 7 divides the real line in 4 disjoint intervals and they are $(-\infty, -5), (-5, 1), (1, 7)\ and\ (7, \infty)$.

For interval $(-\infty,-5)$

g'(y) = 6(y + 5)^2(y – 7)^2(y – 1) < 0

Thus, g(y) is strictly decreasing in $(-\infty,-5)$

For interval (-5,1)

g'(y) = 6(y + 5)^2(y – 7)^2(y – 1) < 0

Thus, g(y) is strictly decreasing in (-5,1).

For interval (1,7)

g'(y) = 6(y + 5)^2(y – 7)^2(y – 1) > 0

Thus, g(y) is strictly increasing in (1,7).

For interval $(7,\infty)$

g'(y) = 6(y + 5)^2(y – 7)^2(y – 1) > 0

Thus, g(y) is strictly increasing in $(7,\infty)$

(iv) Here, g(y) = -2y^3 – 9y^2 – 12y + 1

g'(y) = -6y^2 – 18y – 12

Now, g'(y) = 0

à- 6y^2 – 18y – 12 = 0

à-6(y^2 + 3y + 2) = 0

à-6(y + 2)(y + 1) = 0

à y = -2,-1

Thus, these three points -5, and -1 divides the real line in 3 disjoint intervals and they are $(-\infty,-2),(-2,-1)\ and\ (-1,\infty)$.

For, intervals $(-\infty,-2)\ and\ (-1,\infty)$

g'(y) = -6(y + 2)(y + 1) < 0

Thus, g(y) is strictly decreasing in intervals $(-\infty,-2)\ and\ (-1,\infty)$

For interval (- 2, – 1)

g'(y) = -6(y + 2)(y + 1) > 0

Thus, g(y) is strictly increasing in intervals (- 2, – 1)

(v) Here, g(y) = 7 – 7y – y^2

g'(y) = – 7 – 2y

Now, g'(y) = 0

y = -7/2

Here, the point -7/2 splits the real line into 2 disjoint interval and they are $(-\infty,\frac{-7}{2})\ and\ (\frac{-7}{2},\infty)$

For interval $(-\infty,\frac{-7}{2})$,

g'(y) = – 7 – 2y > 0

Thus, g(y) is strictly increasing in $(-\infty,\frac{-7}{2})$.

For interval $(\frac{-7}{2},\infty)$,

g'(y) = – 7 – 2y< 0

Thus, g(y) is strictly decreasing in $(-\infty,\frac{-7}{2})$.

Q-7) Show that the given function is increasing function throughout its domain

x = $\log(1+y)-\frac{2y}{2+y}$**, y > – 1.**

Ans.)

Here, x = $\log(1+y)-\frac{2y}{2+y}$

$\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{1}{1+y}-\frac{(2+y)(2)-2y(1)}{(2+y)^2}=\frac{1}{1+y}-\frac{4}{(2+y)^2}=\frac{y^2}{(2+y)^2}$

Now, $\frac{\mathrm{d}x}{\mathrm{d}y}=0$

So, $\frac{y^2}{(2+y)^2}=0$

$y^2=0$ (as (2 + y) is not 0; y > -1)

y = 0

As y >- 1, the point y = 0 splits the domain $(-1,\infty)$ in 2 disjoint intervals i.e. (-1,0) and $(0,\infty)$.

For interval (-1,0)

$\frac{\mathrm{d}x}{\mathrm{d}y}$ > 0..................................(a)

Also,

For interval $(0,\infty)$

$\frac{\mathrm{d}x}{\mathrm{d}y}$ > 0..................................(b)

Thus, the given function is increasing throughout its domain.

Q-8) b = [a(a – 2)]^2 is an increasing function , then find out the values of b.

Ans.)

b = [a(a – 2)]^2 = [a^2 – 2a]

b' = 2(a^2 – 2a)(2a – 2) = 4a(a – 1)(a – 2)

Now, b' = 0,

à b = 0,1 and 2

Thus, these three points 0, 1 and 2 divides the real line in 4 disjoint intervals and they are $(-\infty,0),(0,1),(1,2)\ and\ (2,\infty)$.

For intervals $(-\infty,0)\ and\ (1,2)$

b' < 0

Thus, b(a) is strictly decreasing in intervals $(-\infty,0)\ and\ (1,2)$..

For intervals $(0,1)\ and\ (2,\infty)$

b' > 0

Thus, b(a) is strictly increasing in intervals $(0,1)\ and\ (2,\infty)$.

Thus, b(a) is strictly increasing for 0 < a < 1 and a > 2.

Q-9)Show that the given function is increasing function in the domain $[0,\frac{\pi}{2}]$.

f(x) = $\frac{4\sin\alpha}{(2+\cos\alpha)}-\alpha$

Ans.)

Here, f(x) = $\frac{4\sin\alpha}{(2+\cos\alpha)}-\alpha$

f'(x) = $\frac{(2+\cos\alpha)(4\cos\alpha)-4\sin\alpha(-\sin\alpha)}{(2+\cos\alpha)^2}-1$

= $\frac{8\cos\alpha+4\cos^2\alpha+4\sin^2\alpha}{(2+\cos\alpha)^2}-1$

= $\frac{8\cos\alpha+4}{(2+\cos\alpha)^2}-1$

Now, f'(x) = 0

à $\frac{8\cos\alpha+4}{(2+\cos\alpha)^2}-1$ = 0

à $\frac{8\cos\alpha+4}{(2+\cos\alpha)^2}=1$

à$8\cos\alpha+4=4+\cos^2\alpha+4\cos\alpha$

à$\cos^2\alpha-4\cos\alpha=0$

à$\cos\alpha(\cos\alpha-4)=0$

à$\cos\alpha(\cos\alpha-4)=0$

As, $\cos\alpha\neq4,\cos\alpha=0$

$\cos\alpha=0\Rightarrow\cos\alpha=\frac{\pi}{2}$

Now,

$\frac{8\cos\alpha+-(4+\cos^2\alpha+4\cos\alpha)}{(2+\cos\alpha)^2}=\frac{4\cos\alpha-\cos^2\alpha}{(2+\cos\alpha)^2}=\frac{\cos\alpha(4-\cos\alpha)}{(2+\cos\alpha)^2}$

In $[0, \frac{\pi}{2}] \cos \alpha > 0$.

Also, $4 > \cos \alpha$

à$4 - \cos \alpha > 0$

$\cos \alpha (4 - \cos \alpha) > 0$ and $(2 + \cos \alpha)^2 > 0$

à$\frac{\cos \alpha (4 - \cos \alpha)}{(2 + \cos \alpha)^2} > 0$

àf'(x) > 0

Thus, f is strictly increasing in $(0, \frac{\pi}{2})$

Ans as it is also, continuous at x = 0, and x = $\frac{\pi}{2}$

Thus, the given function is increasing function f(x) in the domain $[0, \frac{\pi}{2}]$.

Q-10) Show that $\log y$ is strictly increasing on $(0, \infty)$.

Ans.)

Here, let g(y) = $\log y$.

g'(y) = 1/y

This implies that y > 0,

So, g'(y) > 0.

Thus, $\log y$ is strictly increasing on $(0, \infty)$.

Q-11) Show that the given function is neither strictly increasing nor strictly decreasing on (- 1, 1).

g(y) = 2y^2 – 2y + 1

Ans.)

Here, g(y) = 2y^2 – 2y + 1

g'(y) = 4y – 2

Now, g'(y) = 0

à 4y – 2 = 0

ày = 1/2

Thus, the point ½ splits the interval (- 1, 1) in 2 disjoint intervals and they are (- 1, ½) and (1/2, 1).

For interval (- 1, 1/2)

g'(y) = 4y – 2 < 0.

Thus, g(y) is strictly decreasing in (- 1, 1/2).

For interval (1/2, 1)

g'(y) = 4y – 2 > 0.

Thus, g(y) is strictly increasing in (1/2, 1).

Hence proved.

Q-12) Find out from the following functions that which strictly functions decreasing on $(0, \frac{\pi}{2})$?

(i) $\tan y$

(ii) $\cos 4y$

(iii) $\cos 2y$

(iv) $\sin y$

Ans.)

(i) Here, g(y) = $\tan y$

g'(y) = $\sec^2 y$

For interval $(0, \frac{\pi}{2})$,

g'(y) = $\sec^2 y$> 0

Thus, g(y) is strictly increasing in $(0, \frac{\pi}{2})$.

(ii) Here, g(y) = $\cos 4y$

g'(y) = $-4\sin 4y$

Now, g'(y) = 0,

à$\sin 4y$ = 0

à4y = π; $y \epsilon \left(0, \frac{\pi}{2}\right)$

ày = $\frac{\pi}{4}$

Thus, this point y = $\frac{\pi}{4}$ splits the interval $\left(0, \frac{\pi}{2}\right)$ into 2 disjoint intervals and they are $\left(0, \frac{\pi}{4}\right)$ *and* $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

For interval $\left(0, \frac{\pi}{4}\right)$,

g'(y) = $-4\sin 4y$< 0 ……………$(because 0 < y < \frac{\pi}{4} \Rightarrow 0 < 4y < \pi)$

Thus, g(y) is strictly decreasing in $\left(0, \frac{\pi}{2}\right)$.

For interval $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$,

g'(y) = $-4\sin 4y$> 0 ………$(because \frac{\pi}{4} < y < \frac{\pi}{2} \Rightarrow \pi < 4y < 2\pi)$

Thus, g(y) is strictly increasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

(iii) g(y) = $\cos 2y$

g'(y) = $-2\sin 2y$

Now, $0 < y < \frac{\pi}{2} \Rightarrow 0 < 2y < \pi \Rightarrow \sin 2y > 0 \Rightarrow -2\sin 2y < 0$

Thus, g(y) is strictly decreasing in $\left(0, \frac{\pi}{2}\right)$.

(iv) g(y) = $\sin y$

g'(y) = $\cos y$

For, interval $\left(0, \frac{\pi}{2}\right)$,

g'(y) = $\cos y$ > 0

Thus, g(y) is strictly increasing in $\left(0, \frac{\pi}{2}\right)$.

So, the correct answer is (iii)

Q-13) From the intervals given below, find out in which interval the given function is strictly decreasing?

g(y) = $y^{100} + \sin y - 1$

(i) $\left(0, \frac{\pi}{2}\right)$

(ii) (0, 1)

(iii) $\left(\frac{\pi}{2}, \pi\right)$

(iv) None of above

Ans.)

(iv) None of above

Explanation:

Here, g(y) = $y^{100} + \sin y - 1$

$g'(y)\ 100y^{99} + \cos y$

àFor interval $\left(0, \frac{\pi}{2}\right)$

$\cos y$ > 0, y^{100}> 0

$100y^{99} + \cos y$> 0

Thus, g(y) is strictly increasing in $\left(0, \frac{\pi}{2}\right)$.

à For interval (0, 1)

$\cos y$> 0, y^{100}> 0

$100y^{99} + \cos y$> 0

Thus, g(y) is strictly increasing in (0, 1).

àFor interval $\left(\frac{\pi}{2}, \pi\right)$

$\cos y$> 0, y^{100}> 0

$100y^{99} + \cos y$> 0

Thus, g(y) is strictly increasing in $\left(\frac{\pi}{2}, \pi\right)$

Thus, g(y) is strictly increasing in $(\frac{\pi}{2}, \pi)$.

Thus, the answer is (iv) None of above

Q-14) Find out the least value of m such that given function is strictly increasing on (1, 2).

g(y) = y^2 +my + 1

Ans.)

Here, g(y) = y^2 +my + 1

àg(y) = 2y + m

Now, the g(y) will be increasing on (1, 2) only if,

g'(y) > 0 in (1, 2)

So, g'(y) > 0

à2y + m > 0

à2y > -m

ày = -m/2

Thus, for least value of m,

y > -m/2; $y \epsilon (1, 2)$

à y > -m/2; (1 < y < 2)

Thus, the least requiredvalue of m is

-m/2 = 1

-m/2 = 1

à m = – 2

Thus, the required least value of m is – 2.

Q-15) Assuming D to be any interval disjoint from (-1, 1). Show that the given function is strictly increasing on D.

g(y) = $y + \frac{1}{y}$

Ans.)

Here, g(y) = $y + \frac{1}{y}$

à g'(y) = $1 - \frac{1}{y^2}$

Now, g'(y) = 0

à$\Rightarrow \frac{1}{y^2} = 1 \Rightarrow y = \pm 1$

Thus, y = -1 and 1.

Thus, these three points -1, and 1 divides the real line in 3 disjoint intervals and they are $(-\infty, -1), (-1, 1)\ and\ (1, \infty)$.

For interval (- 1, 1)

-1 < y < 1

à y^2< 1

$\Rightarrow 1 < \frac{1}{y^2}, y \neq 0 \Rightarrow 1 - \frac{1}{y^2} < 0\ on\ (-1, 1) \sim (0)$

Thus, g(y) is strictly decreasing on $(-1, 1) \sim (0)$

For intervals $(-\infty, -1)\ and\ (1, \infty)$

y < -1

à1 < y

ày^2<1

à1 > 1/y^2

à 1 – 1/y^2> 0

Thus, g'(y) > 0

So, g(y) is strictly increasing on $(-\infty, -1)\ and\ (1, \infty)$.

Thus, the given function g(y) is strictly increasing on D which is an interval disjoint of (-1, 1).

Q-16) Show that the given function is strictly increasing on $(0, \frac{\pi}{2})$ and strictly decreasing on $(\frac{\pi}{2}, \pi)$

g(y) = $\log \sin y$

Ans.)

Here, g(y) = $\log \sin y$

g'(y) = $\frac{1}{\sin y} \cos y = \cot y$

For, interval $(0, \frac{\pi}{2})$

g'(y) = cot y > 0

So, g(y) is strictly increasing on $(0, \frac{\pi}{2})$.

For interval $(\frac{\pi}{2}, \pi)$

g'(y) = cot y < 0

So, g(y) is strictly decreasing on$(\frac{\pi}{2}, \pi)$.

Q-17) Show that the given function is strictly decreasing on $(0, \frac{\pi}{2})$ and strictly increasing on $(\frac{\pi}{2}, \pi)$

g(y) = $\log \cos y$

Ans.)

Here, g(y) = $\log \cos y$

g'(y) = $\frac{1}{\cos y}(-\sin y) = -\tan y$

For, interval $(0, \frac{\pi}{2})$

g'(y) = – tan y < 0

So, g(y) is strictly decreasing on $(0, \frac{\pi}{2})$.

For interval $(\frac{\pi}{2}, \pi)$

g'(y) = – tan y > 0

So, g(y) is strictly increasing on $(\frac{\pi}{2}, \pi)$.

Q-18)Show that the given function is increasing in R.

g(y) = $y^3 - 3y^2 + 3y - 100$

Ans.)

Here, g(y) = $y^3 - 3y^2 + 3y - 100$

g'(y) = $3y^2 - 6y + 3$

= $3(y^2 - 2y + 1)$

= $3(y - 1)^2$

Now, for any value of R, $(x - 1)^2 > 0$.

So, g'(y) > 0 for R.

Thus, g(y) is increasing in R.

Q-19) In which of the following interval the function, x = y^2e^{-y}

(i) (-2, 0)

(ii) $(-\infty, \infty)$

(iii) (0, 2)

(iv) $(2, \infty)$

Ans.)

(iii) (0, 2)

Explanation:

Here, x = y^2e^{-y}

So, $\frac{dx}{dy} = 2ye^{-y} - y^{-2}e^{-y} = ye^{-y}(2 - y)$

Now, $\frac{dx}{dy} = 0$

à y = 0 and y = 2

Thus, these two points 0 and 2 divides the real line in 3 disjoint intervals and they are $(-\infty, 0), (0, 2)\ and\ (2, \infty)$.

For intervals $(-\infty, 0)\ and\ (2, \infty)$

g'(y) < 0 (as e^{-y} is always positive)

Thus, g(y) is decreasing on $(-\infty, 0)\ and\ (2, \infty)$.

For interval (0, 2)

g'(y) > 0

Thus, g(y) is increasing on (0, 2).

Thus, the correct answer is (iii).

Excercise 3

Q-1) x = $3y^4$ – 4y is a curve find the slope of the tangent to this curve at y = 4.

Ans.)

Here, x = $3y^4$ – 4y

The slope of the tangent to the curve is

$\frac{dx}{dy} = 12y^3 - 4$

Now, at y = 4

$\frac{dx}{dy}$ = $12(4)^3$ – 4 = 12(64) – 4

= 764

Thus, the slope of the tangent to the curve at y = 4 is 764.

Q-2) x = $\frac{y-1}{y-2}; y \neq 2$ is a curve find the slope of the tangent to this curve at y = 4.

Ans.)

Here, x = $\frac{y-1}{y-2}; y \neq 2$

The slope of the tangent to the curve is

$\frac{dx}{dy} = \frac{(y-2)(1)-(y-1)(1)}{(y-2)^2}$

$= = \frac{y-2-y+1}{(y-2)^2} = \frac{-1}{(y-2)^2}$

Now, y = 4

$\frac{dx}{dy}$ = $\frac{-1}{(4-2)^2} = \frac{-1}{4}$

Thus, the slope of the tangent to the curve at y = 4 is (-1/4).

Q-3) x = y^3 – y + 1 is a curve find the slope of the tangent to this curve at a point where y – co-ordinate is 3.

Ans.)

Here, x = $2y^3$ – y + 1

The slope of the tangent to the curve is

$\frac{dx}{dy}$ = $6y^2$ – 1

Now, at y = 3

$\frac{dx}{dy}$ = $6(3)^2$ – 1 = 6(27) – 1

= 161

Thus, the slope of the tangent to the curve at y = 3 is 161.

Q-4) x = y^3 – 4y + 1 is a curve find the slope of the tangent to this curve at a point where y – co-ordinate is 5.

Ans.)

Here, x = y^3 – 4y + 1

The slope of the tangent to the curve is

$\frac{dx}{dy}$ = $3y^2$ – 4

Now, at y = 3

$\frac{dx}{dy}$ = 3(5)^2 – 4 = 3(25) – 4

= 71

Thus, the slope of the tangent to the curve at y = 5 is 71.

Q-5)x = $a\cos^3 t$ and y = $a\sin^3 t$ are curve so, find the slope of normal to these curves at t = $\frac{\pi}{4}$

Ans.)

Here, x = $a\cos^3 t$ and y = $a\sin^3 t$

$\frac{dx}{dt} = 3a\cos^2 t(-sint) = -3a\cos^2 t\sin t$

Similarly,

$\frac{dy}{dt} = 3a\sin^2 t(\cos t)$

Now,

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3a\sin^2 t(\cos t)}{-3a\cos^2 t\sin t} =- \frac{\sin t}{cost} = -\tan t$

Thus, the slope of the tangent at t = $\frac{\pi}{4}$ is

$\frac{dy}{dx} =- \tan\frac{\pi}{4} =-1$

Now, the slope of the normal at t = $\frac{\pi}{4}$ is

$\frac{-1}{\frac{dy}{dx}} = \frac{-1}{-1} = 1$

Q-6) x = $1 - a\sin t$ and y = $1 - b\cos^2 t$ are curve so, find the slope of normal to these curves at t = $\frac{\pi}{2}$

Ans.)

Here, x = $1 - a\sin t$ and y = $1 - b\cos^2 t$

$\frac{dx}{dt} = -a\cos t$

And

$\frac{dy}{dt} = 2b\cos t(-\sin t) = -2b\sin t\cos t$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2b\sin t\cos t}{-a\cos t} = \frac{2b}{a}\sin t$

Thus, the slope of the tangent at t = $\frac{\pi}{2}$ is

$\frac{dy}{dx} = \frac{2b}{a}\sin\frac{\pi}{2} = \frac{2b}{a}$

Now, the slope of the normal at t = $\frac{\pi}{4}$ is

$\frac{-1}{\frac{dy}{dx}} = \frac{-1}{\frac{2b}{a}} =- \frac{a}{2b}$

Q-7) y = x^3 – 6x^2 -15x +9 is a curve , find points at which the tangent to this curve is parallel to the x- axis.

Ans.)

Here, y = x^3 – 6x^2 -15x +9

$\frac{dy}{dx}$ = 3x^2 – 12x – 15

= 3(x^2 – 4x – 5)

Here, as the tangent is parallel to the x- axis so, its slope is zero.

So,

3(x^2 – 4x – 5) = 0

à(x^2 – 4x – 5) = 0

à(x – 5)(x + 1) = 0

x = 5 or x = – 1

For, x = 3

à(5)^3 – 6(5)^2 -15(5) +9

= – 91

For, x = -1

à$(-1)^3 – 6(-1)^2$ -15(-1) + 9

= 17

Thus, the required points are (5, -91) and (-1, 17).

Q- 8) y = $(x – 2)^2$ is a curve, find points at which the tangent to this curve is parallel to the line joining (2, 0) and (4, 4).

Ans.)

Here, y = $(x – 2)^2$

It is given that the the tangent to the given curve is parallel to the line joining (2, 0) and (4,4). ……………………….(a)

As we know that the parallel lines are having same slope.

So, the slope of the tangent to the curve is

= $\frac{4-0}{4-2} = 4/2 = 2$

Now,

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2(x - x2)$

Now, from (a)

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2$

2(x – 2) = 2

x = 3

For , x = 3

y = $(3 – 2)^2$ = 1

Thus, the required pint is (3, 1).

Q-9) y = x^3 – 11x + 5 is a curve. Find the point on this curve where the tangent is y = x – 11.

Ans.)

Here,

y = x^3 – 11x + 5

$\frac{\mathrm{d}y}{\mathrm{d}x}$ = $3x^2$– 11…………(a)

Here, y = x – 11 is given tangent to the curve.

By comparing it with y = mx + c, we get where m is slope of tangent

So, the slope of tangent to the given curve is 1.

Now, from (a)

$3x^2$ – 11 = 1

à $3x^2$ = 12

àx^2 = 4

à$x = \pm 2$

For, x = -2

y = $(-2)^3$ – 11(-2) + 5 = 19

For, x = 2

y = $(2)^3$ – 11(2) + 5 = – 9

Thus, the required points are (-2, 19) and (2, -9).

Q-10) Find the equation of the all lines with the use of following data:

All the lines are tangent to the curve $y = \frac{1}{x-1},\ x \neq 1$

The slope of the all the lines are same and is -1.

Ans.)

Here, $y = \frac{1}{x-1},\ x \neq 1$

The slope of the tangent to the curve is

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{(x-1)^2}$

The lines are tangent to the given curve and their slope is -1.

So,

$\frac{-1}{(x-1)^2} = -1$

à$(x – 1)^2$ = 1

à(x – 1) = ± 1

àx = 2, 0

For, x = 2

y = $\frac{1}{2-1}$ = 1

For, x = 0

y = $\frac{1}{0-1}$ = – 1

Thus, the 2 tangents will pass through (0, -1) and (2, 1).

The equation of tangent passing through the point (0, -1) is

y – (-1) = -1(x – 0)

ày + 1 = -x

ày + x + 1 = 0

The equation of tangent passing through the point (2, 1) is

y – 1 = -1(x – 2)

ày + 1 = -x + 2

ày + x – 3 = 0

Thus, the required equations of lines are

y + x + 1 = 0

y + x – 3 = 0

Q-11) Find the equation of the all lines with the use of following data:

All the lines are tangent to the curve $y = \frac{1}{x-3},\ x \neq 3$

The slope of the all the lines are same and is 2.

Ans.)

Here, $y = \frac{1}{x-3},\ x \neq 1$

The slope of the tangent to the curve is

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{(x-3)^2}$

The lines are tangent to the given curve and their slope is 2.

So,

$\frac{-1}{(x-3)^2} = 2$

à2(x – 3)^2 = -1

à(x – 3)^2 = -1/2

Which is not possible so, there is no tangent to the given curve having slope 2.

Q-12)Find the equation of the all lines with the use of following data:

All the lines are tangent to the curve $y = \frac{1}{x^2-2x+3}$

The slope of the all the line is same and is 0.

Ans.)

Here, $y = \frac{1}{x^2-2x+3}$

The slope of the tangent to the curve is

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-(2x-2)}{(x^2-2x+3)^2} = \frac{-2(x-1)}{(x^2-2x+3)^2}$

The lines are tangent to the given curve and their slope is 0.

So,

$\frac{-2(x-1)}{(x^2-2x+3)^2} = 0$

à -2(x – 1) = 0

àx = 1

For, x = 1

y = $\frac{1}{1-2+3}$ = ½

Thus, the point is (1, ½)

The equation of tangent with slope zero and passing through the point (1, 1/2) is

y – ½ = 0(x – 1)

ày – ½ = 0

ày = ½

Thus, the equation of the tangent (line) is y = ½.

Q-13) $\frac{x^2}{9} + \frac{y^2}{16} = 1$ **is a curve. Find the point on this curve where the tangent is**

(i) Parallel to y- axis.

(ii) Parallel to x- axis.

Ans.)

Here, $\frac{x^2}{9} + \frac{y^2}{16} = 1$ is a given curve.

Differentiating the given curve w.r.t. 'x'.

$\frac{2x}{9} + \frac{2y}{16} * \frac{\mathrm{d}y}{\mathrm{d}x} = 0$

à$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-16x}{9y}$

(i) For tangent to the curve, parallel to y- axis, slope of the normal is 0.

So,

Slope of normal = $\frac{-1}{\frac{-16x}{9y}} = \frac{9y}{16x} = 0$

ày = 0

Now, for y = 0

à$\frac{x^2}{9} + \frac{(0)^2}{16} = 1$

àx^2 = 9

à$x = \pm 3$

Thus, the tangent is parallel to y- axis at points (-3, 0) and (3, 0).

(ii) For tangent to the curve, parallel to x- axis, slope of the tangent is 0.

So,

$\frac{dy}{dx} = \frac{-16x}{9y} = 0$

à$x = 0$

Now,

à$\frac{(0)^2}{9} + \frac{y^2}{16} = 1$

à$y^2 = 16$

à$y = \pm 4$

Thus, the tangent is parallel to x- axis at points (-4, 0) and (4, 0).

Q-14) Find the equations of the normal and tangent to the curves given below at the points which are indicated in the questions.

a) $y = x^2$ at (0, 0)

b)$y = x^3$ at (1, 1)

c) y = sin c and x = cot c at c = $\frac{\pi}{4}$

d) $y = x^4 – 6x^3 + 13x^2 – 10x + 5$ at (0, 5)

e) $y = x^4 – 6x^3 + 13x^2 – 10x + 5$ at (1, 3)

Ans.)

(a) Here, $y = x^2$ at (0, 0)

Differentiating w.r.t. 'x'

$\frac{dy}{dx} = 2x$

At (0, 0),

$\frac{dy}{dx} = 2(0) = 0$

So, the slope of tangent at (0, 0) is 0.

Now, the equation of tangent

$y – 0 = 0(x – 0)$

à$y = 0$

Now, the slope of normal to the curve is

= $\frac{-1}{Slope\ of\ tangent\ at\ (0,0)} = \frac{-1}{0} = \infty$

Hence not defined

Thus, the equation for normal is

$x = 0$.

(b) Here, $y = x^3$

Differentiating w.r.t. 'x'

$\frac{dy}{dx} = 3x^2$

At (1, 1),

$\frac{dy}{dx} = 3(1)^2 = 3$

So, the slope of tangent at (1, 1) is 3.

Now, the equation of tangent

$y – 1 = 3(x – 1)$

à$y = 3x – 2$

Now, the slope of normal to the curve is

= $\frac{-1}{Slope\ of\ tangent\ at\ (0,0)} = \frac{-1}{3}$

Now, the equation for normal is

y – 1 = (-1/3)(x – 1)

à3y – 3 – 4 = 0

(c) Here, y = sin c and x = cot c

Differentiating w.r.t. 'c'

$\frac{\mathrm{d}y}{\mathrm{d}c} = \cos c \; \frac{\mathrm{d}x}{\mathrm{d}c} = -\sin c \; \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}c}}{\frac{\mathrm{d}x}{\mathrm{d}c}} = \frac{\cos c}{-\sin c} = -\cot c$

At c = $\frac{\pi}{4}$

= $-\cot\frac{\pi}{4} = -1$

So, the slope of tangent at c = $\frac{\pi}{4}$

When c = $\frac{\pi}{4}$ so, x = $\frac{\pi}{4}$ and y = $\frac{\pi}{4}$ is -1.

Now, the equation of the tangent at $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

$$y - \frac{1}{\sqrt{2}} = 1\left(x - \frac{1}{\sqrt{2}}\right)$$
$$\Rightarrow x + y - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$
$$\Rightarrow x + y - \sqrt{2} = 0$$

Now, the slope of normal to the curve is

= $\frac{-1}{\textit{Slope of tangent at } \left(\frac{\pi}{4}, \frac{\pi}{4}\right)} = \frac{-1}{-1} = 1$

Now, the equation for normal is

= $y - \frac{1}{\sqrt{2}} = 1\left(x - \frac{1}{\sqrt{2}}\right)$
$\Rightarrow x = y$

(d) y = $x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5)

Differentiating w.r.t. 'x'

$\frac{\mathrm{d}y}{\mathrm{d}x}$ = $4x^3 - 18x^2 + 26x - 10$

At (0, 5),

$\frac{\mathrm{d}y}{\mathrm{d}x}$ = $4(0)^3 - 18(0)^2 + 26(0) - 10 = -10$

So, the slope of tangent at (0, 5) is -10.

Now, the equation of tangent

y – 5 = -10(x – 0)

à10x + y – 5 = 0

Now, the slope of normal to the curve is

= $\frac{-1}{\textit{Slope of tangent at } (0,5)} = \frac{-1}{-10} = \frac{1}{10}$

Now, the equation for normal is

y – 5 = (1/10)(x – 0)

à10y – 50 = x

àx – 10y = – 50

(e) y = $x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5)

Differentiating w.r.t. 'x'

$\frac{\mathrm{d}y}{\mathrm{d}x}$ = $4x^3 - 18x^2 + 26x - 10$

At (1, 3),

$\frac{\mathrm{d}y}{\mathrm{d}x}$ = $4(1)^3 - 18(1)^2 + 26(1) - 10 = 2$

So, the slope of tangent at (1, 3) is 2.

Now, the equation of tangent

y – 3= 2(x – 1)

à2x – y +1 = 0

Now, the slope of normal to the curve is

$= \frac{-1}{Slope\ of\ tangent\ at\ (1,3)} = \frac{-1}{2}$

Now, the equation for normal is

y – 3= (-1/2)(x – 1)

à2y – 6 = – x + 1

àx + 2y = 7

Q-15) y = x^2 – 2x + 7 is a curve. Find the equation to the tangent line to the given curve which is:

(i) Perpendicular to line -15x + 15y -10 = 0.

(ii) Parallel to line 2x – y + 9 = 0.

Ans:

Here, y = x^2 – 2x + 7.

Differentiating w.r.t. 'x',

$\frac{dy}{dx}$ = 2x – 2……………..(a)

(i) The given curve is perpendicular to the line -15x + 5y = 13.

i.e. y = 3x + 13/5

By comparing it with y = mx + c, we get

Slope of the line = 3

The given curve is perpendicular to line y = 3x + 13/5, so the slope of the curve is given by,

$\Rightarrow \frac{-1}{slope\ of\ the\ line} = \frac{-1}{3}$

à2x – 2 = -(1/3)

à2x = 2 – (1/3)

à x = 5/6

Now, for x = 5/6,

y = $(5/6)^2$ – 2(5/6) + 7 = 217/36

Thus, the tangent is passing through the point (5/6, 217/36), so its equation is given by

$y - \frac{217}{36} = \frac{-1}{3}(x - \frac{5}{6})$

à$\frac{36y-217}{36} = \frac{-1}{18}(6x-5)$

à36y – 217 = -2((6x – 5)

à36y – 217 = -12x + 10

à36y – 12x – 227 = 0

Hence the required equation of tangent is 36y – 12x – 227 = 0.

(ii) The given curve is parallel to the line 2x – y + 9 = 0.

i.e. y = 2x + 9

By, comparing it with y = mx + c, we get

Slope of the line = 2

As the given curve is parallel to line y = 2x + 9. So, their slope will be equal.

Now, from (a)

2x – 2 = 2

àx = 2

For, x = 2,

y = $(2)^2$ – 2(2) + 7= 7

Thus, the tangent is passing through the point (2, 7), so its equation is given by

y – 7 = 2(x – 2)

ày = 2x – 3 = 0

Hence the required equation of tangent is y = 2x – 3 = 0.

Q-16) $y = 7x^3 + 11$ is a curve, show that the tangent to this curve at x = -2 and x = 2 are parallel.

Ans.)

Here, $y = 7x^3 + 11$

$\frac{dy}{dx} = 21x^2$

Thus, the slope of tangent to the curve is $21x^2$.

Now, the slope of the tangent at x = -2 is

$= 21(-2)^2 = 21*4 = 84$

The slope of the tangent at x = 2 is

$= 21(2)^2 = 21*4 = 84$

Here the slope of both the tangent is equal. So, the two tangents are parallel.

Q-17) y = x3 is a curve, find the points on the curve at which the slope of the tangent is equal to the y –co-ordinate of the point.

Ans.)

Here, $y = x^3$

$\frac{dy}{dx} = 3x^2$

Thus, the slope of tangent to the curve is $3x^2$.

Now, the slope of the tangent at (x, y) is

$= 3(x)^2$

Now it is given that slope of the tangent is equal to the y –co-ordinate of the point.

$3x^2 = y \; 3x^2 = x^3 \; x^2(x-3) = 0$

x = 3, x = 0.

For, x = 0

$y = (0)^3 = 0$

For, x = 3

$y = (3)^3 = 9$

Thus, the required points are (0, 0) and (3, 9).

Q-18) $y = 4x^3 – 2x^5$is a curve, find all the points at which the tangent passes through (0, 0).

Ans.)

Here, $y = 4x^3 – 2x^5$

$\frac{dy}{dx} = 12x^2 – 10x^4$

Thus, the slope of tangent to the curve is $12x^2 – 10x^4$.

Now, the slope of the tangent at (x, y) is

$Y – y = (12x^2 – 10x^4)(X – x)$……………………………(a)

Here, the tangent passes through (0,0), So X = 0 and Y = 0.

Thus, from (a)

$-y = (12x^2 – 10x^4)(-x)$

à$y = 12x^3 – 10x^5$

à$4x^3 – 2x^5 = 12x^3 – 10x^5$

à$8x^5 – 8x^3 = 0$

à$x^5 – x^3 = 0$

à$x^3(x^2 – 1)= 0$

à$x = 0 \; or \; x = \pm 1$

Now, for x = 0

$y = 4(0)^3 – 2(0)^5 = 0$

For, x = -1

y = $4(-1)^3 - 2(-1)^5 = -2$

For, x = 1

y = $4(1)^3 - 2(1)^5 = 2$

Thus, the required points are (0,0), (-1,-2) and (1,2).

Q-19) $x^2 + y^2 - 2x - 3 = 0$ is a curve. Find the points at which the tangent is parallel to x- axis.

Ans.)

Here, $x^2 + y^2 - 2x - 3 = 0$

Differentiating w.r.t. 'x',

$$2x + 2y\frac{dy}{dx} - 2 = 0$$
$$\Rightarrow y\frac{dy}{dx} = 1 - x$$
$$\Rightarrow \frac{dy}{dx} = \frac{1-x}{y}$$

Here, as the tangent is parallel to the x- axis. So, the slope of the tangent is 0.

$$\frac{1-x}{y} = 0$$
$$\Rightarrow 1 - x = 0$$
$$\Rightarrow x = 1$$

For, x = 1

$(1)^2 + y^2 - 2(1) - 3 = 0$

à$y^2 = 4$

à$y = \pm 2$

Thus, the required points are (1, -2) and (1,2).

Q-20) $ty^2 = x^3$ is a curve. Find the equation of normal to this curve at (tn^2, tn^3).

Ans.)

Here, $ty^2 = x^3$

Differentiating w.r.t. 'x'

$$2ty\frac{dy}{dx} = 3x^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{2ty}$$

The slope of tangent at (tn^2, tn^3) is

$$\frac{dy}{dx} = \frac{3(tn^2)^2}{2t(tn^3)} = \frac{3t^2n^4}{2t^2n^3} = \frac{3n}{2}$$

Now, the slope of normal at (tn^2, tn^3) is given by

$$= \frac{-1}{Slope\ of\ tangent\ at\ (tn^2, tn^3)} = \frac{-2}{3n}$$

Thus, the equation of the normal at(tn^2, tn^3) is

$y - tn^3 = \frac{-2}{3n}(x - tn^2)$

à$3ny - 3tn^4 = -2x + 2n^2$

à$2x + 3ny - tn^2(2 + 3n^2) = 0$

Q-21) $y = x^3 + 2x + 6$ is a curve. Find the equation of normal to this curve and the normal is parallel to the lone $x + 14y + 4 = 0$.

Ans.)

Here, $y = x^3 + 2x + 6$

$\frac{dy}{dx} = 3x^3 + 2$

Now, the slope of tangent at(x, y) is

$= 3x^3 + 2$

Thus, the slope of normal at (x, y) is given by

$$= \frac{-1}{Slope\ of\ tangent\ at\ (x, y)} = \frac{-1}{3x^2 + 2}$$

The normal to the curve is parallel to the line x + 14y + 4 = 0, so slope for both of them is equal.

Thus, the slope of line x + 14y + 4 = 0 is $\frac{-1}{3x^2+2}$..........(a)

Now, x + 14y + 4 = 0 can be written as

y = – (x/14) – (7/2)

On comparing above equation with y = mx + c, we get

Slope of line x + 14y + 4 = 0 equals to (-1/14).

From (a)

$\frac{-1}{3x^2+2} = \frac{-1}{14}$

à$3x^2 + 2 = 14$

à$3x^2 = 12$

à$x^2 = 4$

àx = -2 or x = 2

For, x = -2

$y = (-2)^3 + 2(-2) + 6 = -6$

For, x = 2

$y = (2)^3 + 2(2) + 6 = 18$

Thus, there exits 2 normal to the given curve with same slope of (-1/14) and pasing through two distinct points (-2, -16) and (2,18).

The equation of normal passing through (-2, -16) is

y – (-6) = (-1/14)(x – (-2))

ày +6 = (-1/14)(x + 2)

à14y + 84= -x – 2

àx +14y = -86

The equation of normal passing through (2, 18) is

y – 18 = (-1/14)(x – 2)

à14y – 252 = -x + 2

à x + 14y = 254

Thus, the required equations of the normal are

àx +14y = -86

àx +14y = 254

Q-22) Parabola y^2 = 4mxis given. Find the equations for normal and tangent to this parabola at (mt^2, 2mt).

Ans.)

Here, $y^2 = 4mx$

Differentiating w.r.t. 'x'

$2y\frac{dy}{dx} = 4m$

à$\frac{dy}{dx} = \frac{2m}{y}$

Now, the slope of tangent at(mt^2, 2mt) is

= 2m/2mt = 1/t

Now, the equation of tangent to the given parabola at (mt^2, 2mt) is given by,

$y – 2mt = 1/t(x – mt^2)$

à$ty – 2mt^2 = x – mt^2$

à$ty = x + mt^2$

Now, the slope of normal to the parabola at (mt^2, 2mt) is given by,

$\frac{-1}{Slope\ of\ tangent\ at\ (mt^2,2mt)} = -t$

Thus, the equation of the normal to parabola at (mt^2, 2mt) is

y – 2mt = -t(x – mt^2)

ày – 2mt = -tx + mt^3

ày = -tx + 2mt + mt^3

Q-23)Show that the curves xy = m and x = y^2 cut at 90 degrees if $8m^2$ = 1.

[Hint: 2 curves intersect at 90 degrees if the tangent to those 2 curves are perpendicular to each other at the point of intersection.]

Ans.)

Here, xy = m and x = y^2

à(y^2)y = m

ày^3 = m

ày = $m^{1/3}$

Thud, x = $m^{2/3}$

Hence the point of intersection of curves is ($m^{2/3}$, $m^{1/3}$)

Now, x = y^2

Differentiating w.r.t. 'x'

$$1 = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2y}$$

Now, slope of tangent to the curve x = y^2at ($m^{2/3}$, $m^{1/3}$) is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2m^{1/3}}$$

Now, xy = m

Differentiating w.r.t. 'x'

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y}{x}$$

Now, slope of tangent to the curve xy = m at ($m^{2/3}$, $m^{1/3}$) is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-m^{1/3}}{m^{2/3}} = \frac{-1}{m^{1/3}}$$

As we know that, 2 curves intersect at 90 degrees if the tangent to those 2 curves are perpendicular to each otherat the point of intersection.

i.e. the product of slope of both the tangent is -1.

So, $(\frac{1}{2m^{2/3}})(\frac{-1}{m^{1/3}}) = -1$

à$2m^{2/3}$ = 1

à$(2m^{2/3})^3$ = $(1)^3$

à$8m^2$ = 1

Hence proved.

Q-24) Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given. Find the equations of tangent and normal to this hyperbola at point (m, n).

Ans.)

Here, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Differentiating w.r.t. 'x'

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\Rightarrow \frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{a^2}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b^2x}{a^2y}$$

Now, the slope of tangent at (m, n) is

$= \frac{b^2m}{a^2n}$

The equation of tangent at (m, n) is

$y-n=\frac{b^2m}{a^2n}(x-m) \Rightarrow a^2yn-a^2n^2=b^2xm-b^2m^2$
$\Rightarrow b^2xm-b^2m^2-a^2yn+a^2n^2=0$

$\Rightarrow \frac{xm}{a^2}-\frac{yn}{b^2}-(\frac{m^2}{a^2}-\frac{n^2}{b^2})=0$ (dividing both sides by a^2b^2)

$\Rightarrow \frac{xm}{a^2}-\frac{yn}{b^2}-1=0$

(as (m, n) lies on the hyperbola)

$\Rightarrow \frac{xm}{a^2}-\frac{yn}{b^2}=1$

The slope of normal at (m, n) is given by,

$\frac{-1}{Slope\ of\ tangent\ at\ (m,n)}=\frac{-a^2n}{b^2m}$

Now, the equation of normal at (m, n) is

$y-n=\frac{-a^2n}{b^2m}(x-m)$
$\Rightarrow \frac{y-n}{a^2n}=\frac{-(x-m)}{b^2m}=0$
$\Rightarrow \frac{y-n}{a^2n}+\frac{x-m}{b^2m}=0$

Q-25)$y=\sqrt{3x-2}$ is a curve. Find the equations of the tangent to the curve which is parallel to 4x – 2y+ 5 = 0.

Ans.)

Here, $y=\sqrt{3x-2}$

$\frac{dy}{dx}=\frac{3}{2\sqrt{3x-2}}$

Now, the slope of the tangent at (x, y) is

$=\frac{3}{2\sqrt{3x-2}}$

Now, 4x – 2y + 5 = 0 can be written as

y = 2x + 5/2

Comparing y = 2x + 5/2 with y = mx + c , we get

Slope of line = 2

Now the tangent is parallel to the line 4x – 2y + 5 = 0. So,

Slope of the tangent = Slope of line 4x – 2y+ 5 = 0

$\frac{3}{2\sqrt{3x-2}}=2$
$\Rightarrow \sqrt{3x-2}=3/4$
$\Rightarrow 3x-2=9/16$
$\Rightarrow 3x=9/16+2=41/6$
$\Rightarrow x=41/48$

For, x = 41/48

$y=\sqrt{3(\frac{41}{48})-2}=\sqrt{\frac{41}{16}-2}=\sqrt{\frac{41-32}{16}}=\sqrt{\frac{9}{16}}=3/4$

Now, the equation of tangent at $(\frac{41}{48},\frac{3}{4})$ is

y – ¾= 2(x -41/48)

$y-\frac{3}{4}=2(x-\frac{41}{48})$
$\Rightarrow \frac{4y-3}{4}=2$
$\Rightarrow 4y-3=\frac{48x-41}{6}$
$\Rightarrow 24y-18=48x-41$
$\Rightarrow 48x-24y=23$

Thus, the required equation is 48x -24y -23 = 0.

Q-26) $y=2x^3+4\sin x$ is a curve. Find the slope of normal to the curve at x = 0.

(i) -4

(ii) -1/4

(iii) 1/4

(iv) 4

Ans.)

(ii) -1/4

Explanation:

Here, $y = 2x^3 + 4\sin x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x + 4\cos x$

Now the slope of tangent at x = 0 is

= $6(0) + 4\cos(0) = 4$

Now, the slope of the normal to the curve is given by

= $\frac{-1}{Slope\ of\ tangent\ at\ x=1} = \frac{-1}{4}$

Q-27) The line x –y + 1 = 0 is a tangent to the parabola y^2 = 4x at point ____.

(i) (2, 1)

(ii) (1, -2)

(iii) (1, 2)

(iv) (-1, 2)

Ans.)

(iii) (1, 2)

Explanation:

Here, y^2 = 4x

For finding the slope of tangent

Differentiating w.r.t. 'x'

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4$ ……………..(a)
$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{y}$

Here, the tangent to the curve is x – y + 1 = 0, which can be written as y = x + 1 and comparing it with y = mx + c, we get

Slope of tangent = 1

From (a)

2/y = 1

ày = 2

For y = 2

2 = x + 1

àx = 1

Thus, the required point is (1, 2).

Exercise 4 :

Q-1) Find the approximate value up to 3 decimal places using differentials

(a)$(32.15)^{\frac{1}{5}}$

(b) $(3.968)^{\frac{3}{2}}$

(c) $(81.5)^{\frac{1}{4}}$

(d) $(26.57)^{\frac{1}{3}}$

(e) $\sqrt{0.0037}$

(f) $(401)^{\frac{1}{2}}$

(g) $(82)^{\frac{1}{4}}$

(h) $(255)^{\frac{1}{4}}$

Ans:

(a) $(32.15)^{\frac{1}{5}}$

1

Let, $y = x^{\frac{1}{5}}$.

Let x = 32 and $\Delta x = 0.15$

Now,

$$\Delta y = (x + \Delta x)^{\frac{1}{5}} - x^{\frac{1}{5}} = (32.15)^{\frac{1}{5}} - 32^{\frac{1}{5}} = (32.15)^{\frac{1}{5}} - 2$$
$$\Rightarrow (32.15)^{\frac{1}{5}} = 2 + \Delta y$$

Also, dy is approximately equal to Δy, so we get

$dy = \frac{\mathrm{d}y}{\mathrm{d}x}\Delta x = \frac{1}{5(x)^{\frac{4}{5}}}.(\Delta x)$ (as $y = x^{\frac{1}{5}}$)

$$= \frac{1}{5*(2)^4}.(0.15)$$
$$= \frac{0.15}{80} = 0.00187$$

Thus, the approximate value of $(32.15)^{\frac{1}{5}}$ is 2 + 0.00187 = 2.00187.

(b) $(3.968)^{\frac{3}{2}}$

Let, $y = x^{\frac{3}{2}}$.

Let x = 4 and $\Delta x = -0.032$

Now,

$$\Delta y = (x + \Delta x)^{\frac{3}{2}} - x^{\frac{3}{2}} = (3.968)^{\frac{3}{2}} - 4^{\frac{3}{2}} = (3.968)^{\frac{3}{2}} - 8$$
$$\Rightarrow (3.968)^{\frac{3}{2}} = 8 + \Delta y$$

Also, dy is approximately equal to Δy, so we get

$dy = \frac{\mathrm{d}y}{\mathrm{d}x}\Delta x = \frac{3}{2}(x)^{\frac{1}{2}}.(\Delta x)$ (as $y = x^{\frac{3}{2}}$)

= 1.5(2)(-0.032)

= -0.096

Thus, the approximate value of $(3.968)^{\frac{3}{2}}$ is 8 -0.096 = 7.904.

(c) $(81.5)^{\frac{1}{4}}$

Let, $y = x^{\frac{1}{4}}$.

Let x = 81 and $\Delta x = 0.5$

Now,

$$\Delta y = (x + \Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}} = (81.5)^{\frac{1}{4}} - 81^{\frac{1}{4}} = (81.5)^{\frac{1}{4}} - 3$$
$$\Rightarrow (81.5)^{\frac{1}{4}} = 3 + \Delta y$$

Also, dy is approximately equal to Δy, so we get

$dy = \frac{\mathrm{d}y}{\mathrm{d}x}\Delta x = \frac{1}{4*(x)^{\frac{3}{4}}}.(\Delta x)$ (as $y = x^{\frac{1}{4}}$)

$$= \frac{1}{4*27}.(0.5)$$
$$= \frac{0.5}{108} = 0.0046$$

Thus, the approximate value of $(81.5)^{\frac{1}{4}}$ is 3 + 0.0046 = 3.0046.

(d) $(26.57)^{\frac{1}{3}}$

Let, $y = x^{\frac{1}{3}}$.

Let x = 27 and $\Delta x = -0.43$

Now,

$$\Delta y = (x + \Delta x)^{\frac{1}{3}} - x^{\frac{1}{3}} = (26.57)^{\frac{1}{3}} - 27^{\frac{1}{3}} = (26.57)^{\frac{1}{3}} - 3$$
$$\Rightarrow (26.57)^{\frac{1}{3}} = 3 + \Delta y$$

Also, dy is approximately equal to Δy, so we get

$dy = \frac{\mathrm{d}y}{\mathrm{d}x}\Delta x = \frac{1}{3*(x)^{\frac{2}{3}}}.(\Delta x)$ (as $y = x^{\frac{1}{3}}$)

$$= \frac{1}{3*(9)}(-0.43)$$
$$= \frac{-0.43}{27} = -0.015$$

Thus, the approximate value of $(26.57)^{\frac{1}{3}}$ is 3 - 0.015 = 2.985

Thus, the approximate value of $(26.57)^{\frac{1}{3}}$ is 3 + -0.015 = 2.985.

(e) $\sqrt{0.0037}$

Let, $y = x^{\frac{1}{2}}$.

Let x = 0.0036 and $\Delta x = 0.0001$

Now,

$$\Delta y = (x + \Delta x)^{\frac{1}{2}} - x^{\frac{1}{2}} = (0.0037)^{\frac{1}{2}} - (0.0036)^{\frac{1}{2}} = (0.0037)^{\frac{1}{2}} - 0.06$$
$$\Rightarrow (0.0037)^{\frac{1}{2}} = 0.06 + \Delta y$$

Also, dy is approximately equal to Δy, so we get

$dy = \frac{\mathrm{d}y}{\mathrm{d}x}\Delta x = \frac{1}{2*(x)^{\frac{1}{2}}}.(\Delta x)$ (as $y = x^{\frac{1}{2}}$)

$$= \frac{1}{2*(0.06)}(0.0001)$$
$$= \frac{0.0001}{0.12} = 0.00083$$

Thus, the approximate value of $(0.0037)^{\frac{1}{2}}$ is 0.06 + 0.00083 = 0.06083.

(f) $(401)^{\frac{1}{2}}$

Let, $y = x^{\frac{1}{2}}$.

Let x = 400 and $\Delta x = 1$

Now,

$$\Delta y = (x + \Delta x)^{\frac{1}{2}} - x^{\frac{1}{2}} = (401)^{\frac{1}{2}} - (400)^{\frac{1}{2}} = (401)^{\frac{1}{2}} - 20$$
$$\Rightarrow (401)^{\frac{1}{2}} = 20 + \Delta y$$

Also, dy is approximately equal to Δy, so we get

$dy = \frac{\mathrm{d}y}{\mathrm{d}x}\Delta x = \frac{1}{2*(x)^{\frac{1}{2}}}.(\Delta x)$ (as $y = x^{\frac{1}{2}}$)

$$= \frac{1}{2*(20)}(1)$$
$$= \frac{1}{40} = 0.025$$

Thus, the approximate value of $(401)^{\frac{1}{2}}$ is 20 + 0.025 = 20.025.

(g) $(82)^{\frac{1}{4}}$

Let, $y = x^{\frac{1}{4}}$.

Let x = 81 and $\Delta x = 1$

Now,

$$\Delta y = (x + \Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}} = (82)^{\frac{1}{4}} - 81^{\frac{1}{4}} = (82)^{\frac{1}{4}} - 3$$
$$\Rightarrow (82)^{\frac{1}{4}} = 3 + \Delta y$$

Also, dy is approximately equal to Δy, so we get

$dy = \frac{\mathrm{d}y}{\mathrm{d}x}\Delta x = \frac{1}{4*(x)^{\frac{3}{4}}}.(\Delta x)$ (as $y = x^{\frac{1}{4}}$)

$$= \frac{1}{4*27}.(1)$$
$$= \frac{1}{108} = 0.009$$

Thus, the approximate value of $(82)^{\frac{1}{4}}$ is 3 + 0.009 = 3.009.

(h) $(255)^{\frac{1}{4}}$

Let, $y = x^{\frac{1}{4}}$.

Let x = 256 and $\Delta x = -1$

Now,

$$\Delta y = (x + \Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}} = (255)^{\frac{1}{4}} - 256^{\frac{1}{4}} = (255)^{\frac{1}{4}} - 4$$
$$\Rightarrow (255)^{\frac{1}{4}} = 5 + \Delta y$$

Also, dy is approximately equal to Δy, so we get

$dy = \frac{\mathrm{d}y}{\mathrm{d}x}\Delta x = \frac{1}{4*(x)^{\frac{3}{4}}}.(\Delta x)$ (as $y = x^{\frac{1}{4}}$)

$= \frac{1}{4*256^{\frac{3}{4}}}.(-1)$
$= \frac{-1}{4*64} = -0.0039$

Thus, the approximate value of $(255)^{\frac{1}{4}}$ is 4 – 0.0039 = 3.9961.

Excercise 5 :

Que.1. For the given functions, find the maximum and minimum values:

(i) f(y) = $(2y – 1)^2$ + 3

Ans. f(y) = $(2y – 1)^2 + 3$

It is observed that $(2y – 1)^2 \geq 0$ for y ϵ R.

Hence, f(y) = $(2y – 1)^2 + 3 \geq 3$ for y ϵ R.

When 2y – 1 = 0, the minimum value of f is obtained.

2y – 1 = 0. Therefore, x = $\frac{1}{2}$

Minimum value = f($\frac{1}{2}$) = (2. $\frac{1}{2}$ – 1 $)^2$ + 3 = 3

Therefore, the maximum value of the function f does not exist.

(ii)f(y) = $9y^2$ + 12y + 2 = $(3y + 2)^2$ – 2.

Ans. It is observed that $(3y + 2)^2 \geq 0$ for y ϵ R.

Hence, f(y) = $(3x + 2)^2 – 2 \geq –2$ for y ϵ R.

When 3y + 2 = 0, the minimum value of f is obtained.

3y + 2 = 0. Therefore, y = $\frac{-2}{3}$

Minimum value = f($\frac{-2}{3}$) = (3 ($\frac{-2}{3}$) + 2$)^2$ – 2 = –2

Therefore, the maximum value of the function f does not exist.

(iii) f(y) = $–(y – 1)^2$ + 10.

Ans. It is observed that $(x – 1)^2 \geq 0$ for y ϵ R.

Hence, f(y) = $–(y – 1)^2 + 10 \leq 10$ for y ϵ R.

When (y – 1) = 0, the maximum value of f is obtained.

(y – 1) =0. Therefore, y = 0

Maximum value = f(1) = $–(1 – 1)^2$ + 10 = 10

Therefore, the minimum value of the function f does not exist.

(iv) g(y) = y^3 + 1

Ans. Therefore, the function g does not have maximum or minimum value.

Que.2. For the given functions, find the maximum and minimum values:

(i) f(y) = $|y+2|$ – 1

Ans. $|y+2| \geq 0$ for y ϵ R

Hence, f(y) = $|x+2| – 1 \geq – 1$ for y ϵ R

When $|x+2|$ =0, the minimum value of f is obtained.

$\Rightarrow$ x = –2

Minimum value = f(–2) = $|-2+2|$ –1 = –1

Therefore, the maximum value of the function f does not exist.

(ii) g(y) = – $|y+1|$ + 3

Ans. – $|y+1| \leq 0$ for y ϵ R

g(x) = – $|y+1|$ + 3 for y ϵ R

When $|y+1|$ = 0, the maximum value of g is obtained.

x = –1

Maximum value = g(–1) = – $|-1+1|$ + 3 = 3

Maximum value = g(-1) = $-|-1+1|$ + 3 = 3

Therefore, the minimum value of the function f does not exist.

(iii) h(y) = sin 2y + 5

Ans. – 1 $\leq$ sin 2y $\leq$ 1.

–1 + 5 $\leq$ sin 2y + 5 $\leq$ 6

Therefore, 6 and 4 are the maximum and minimum values.

(iv) f(y) = $|sin4y+3|$

Ans. –1 $\leq$ sin 4y $\leq$ 1

2 $\leq$ sin 4y + 3 $\leq$ 4

2 $\leq$ $|sin4y+3|$ $\leq$ 4

Therefore 4 and 2 are the maximum and minimum values.

(v) h(y) = y + 1, y (-1,1)

Ans. If point y_0 is close to – 1, then $\frac{y_0}{2}$ $<$ y_0 + 1 for y ϵ (-1, 1)

If y_1 is close to 1, then y_1 + 1 < $\frac{y_1+1}{2}$ + 1 for y ϵ (-1,1).

Therefore, h(y) neither has a maximum or a minimum value at (-1,1).

Que.3. For the following, find the local maxima and minima and also local maximum and minimum values, if any.

(i) f(y) = y^2

$f'(y)$(x) = 2y

$f"(y)$ = 0 $\Rightarrow$ y = 0

Thus, the critical point y = 0 can be the local maxima or local minima of f.

$f"(y)$ = 2 which is positive

Thus, by second derivative test, y = 0 is the point of local maxima and minima value of f at y = 0 is f(0) = 0.

(ii) g(y) = y^3 – 3y

g'(y) = 3y^2 – 3

g'(y) = 0

3y^2 = 3

y = $\pm$1

$g"$(y) = 6y

$g"$(1) = 6 > 0

g'(-1) = – 6 < 0

By second derivative,

The point of local minimum and minima at g is y = 1

g(1) = 1^3 – 3 = 1 – 3 = –2

The point of local maximum and local maxima at g is y = – 1

g(-1) = $(-1)^3$ – 3 (– 1) = – 1 + 3 = 2.

(iii) h(y) = sin y + cos y, 0 < y < $\frac{\pi}{2}$

h'(y) = cos y – sin y

h'(y) = 0

Sin y = cos y

tan y = 1 $\Rightarrow$ y = $\frac{\pi}{4}$$\epsilon$ (0, $\frac{\pi}{2}$)

$h"$(y) = – sin y – cos y = –(sin y + cos y)

$h"$ $\frac{\pi}{4}$ = –($\frac{1}{\sqrt{2}}$ + $\frac{1}{\sqrt{2}}$) = $-\frac{2}{\sqrt{2}}$ = $-\sqrt{2}$ < 0

Thus, by second derivative, the local maximum and maxima at point y = $\frac{\pi}{4}$ is

h($\frac{\pi}{4}$) = sin $\frac{\pi}{4}$ + cos $\frac{\pi}{4}$ = $\frac{1}{\sqrt{2}}$ +$\frac{1}{\sqrt{2}}$ = $\sqrt{2}$.

(iv) f(y) = sin y – cos y, 0 < y < 2n

f'(y) = cos y + sin y

h'(y) = 0

cos y = – sin y

tan y = – 1 ⇒ y = $\frac{3\pi}{4}$, $\frac{7\pi}{4}$ ϵ (0,22π)

f”(y) = – sin y + cos y

f”($\frac{3\pi}{4}$) = – sin $\frac{3\pi}{4}$ + cos $\frac{3\pi}{4}$ = $-\frac{1}{\sqrt{2}}$ – $\frac{1}{\sqrt{2}}$ = $-\sqrt{2}$ > 0

f”($\frac{7\pi}{4}$) = – sin $\frac{7\pi}{4}$ + cos $\frac{7\pi}{4}$ = $\frac{1}{\sqrt{2}}$ + $\frac{1}{\sqrt{2}}$ = $\sqrt{2}$ > 0

Thus, by second derivative, the local maximum and maxima at point y = $\frac{3\pi}{4}$ is

f($\frac{3\pi}{4}$ = sin $\frac{3\pi}{4}$ – cos $\frac{3\pi}{4}$ = $\frac{1}{\sqrt{2}}$ + $\frac{1}{\sqrt{2}}$ = $\sqrt{2}$

The local minimum and minima at point y = $\frac{7\pi}{4}$ is

sin $\frac{7\pi}{4}$ – cos $\frac{7\pi}{4}$ = $-\frac{1}{\sqrt{2}}$ – $\frac{1}{\sqrt{2}}$ = $-\sqrt{2}$

(v) f(y) = y^3 – $6y^2$ + 9y + 15

f'(y) = $3y^3$ – 12y + 9

f'(y) = 0

3(y^2 – 4y + 3) = 0

3(y – 1) (y – 3) = 0

y = 1,3

f”((y) = 6y – 12 = 6(x – 2)

f”(= 6(1 – 2) = -6 < 0

f”$'$(= 6(3 – 2) = 6 > 0

Thus, by second derivative, the local maximum and maxima at point y = 1 is

f(1) = 1 – 6 + 9 + 15 = 19

The local minimum and minima at point y = 3 is

f(3) = 27 – 54 + 27 + 15 = 15

(vi) g(y) = $\frac{y}{2}$ + $\frac{2}{y}$, y > 0

g'(y) = $\frac{1}{2}$ – $\frac{2}{y^3}$

g'(y) = 0

$\frac{2}{y^3}$ = $\frac{1}{2}$

y^3 = 4 ⇒ y = $\pm$2

As y > 0, y = 2

g”(y) = $\frac{4}{y^3}$

g”(2) = $\frac{4}{2^3}$ = $\frac{1}{2}$ > 0

By second derivative, the local minimum and minima at point y =2 is

g(2) = $\frac{2}{2}$ +$\frac{2}{2}$ = 1+ 1 = 2.

(vii) g(y) = $\frac{1}{y^2+2}$

g'(y) = $\frac{-2y}{(y^2+2)^2}$

g'(y) = 0

$\frac{-2y}{(y^2+2)^2}$ = 0

y = 0

By first derivative, the local maximum and maxima at point y = 0 is

g(0) = $\frac{1}{0+2}$ = $\frac{1}{2}$.

(viii) f(y) = y$\sqrt{1-y}$, y > 0

f'(y) = $\sqrt{1-y}$ + y . $\frac{1}{2\sqrt{1-y}}$ (-1) = $\sqrt{1-y}$ – $\frac{y}{2\sqrt{1-y}}$ = $\frac{2-3y}{2\sqrt{1-y}}$

f'(y) = 0

$\frac{2-3y}{2\sqrt{1-y}}$ = 0 ⇒ 2 – 3y

y = $\frac{2}{3}$

f"(y) = $\frac{1}{2}$ $\frac{\sqrt{1-y}(-3)-(2-3y)(\frac{-1}{2\sqrt{1-y}})}{1-y}$

= $\frac{\sqrt{1-y}(-3)-(2-3y)(\frac{-1}{2\sqrt{1-y}})}{2(1-y)}$

= $\frac{-6(1-y)+(2-3y)}{4(1-y)^{\frac{3}{2}}}$

=$\frac{3y-2}{4(1-y)^{\frac{3}{2}}}$

f"(y) = $\frac{2-4}{4(\frac{1}{3})^{\frac{3}{2}}}$ =$\frac{-1}{2(\frac{1}{3})^{\frac{3}{2}}}$ < 0

By second derivative, the local maximum and maxima at point y = $\frac{2}{3}$ is

f($\frac{2}{3}$) = $\frac{2}{3}$ $\sqrt{1-\frac{2}{3}}$ = $\frac{2}{3\sqrt{3}}$ = $\frac{2\sqrt{3}}{9}$.

Que.4. Show that the given equation does not have maxima and minima:

f(y) = e^y

Ans.

f'(y) = e^y

f'(y) = 0

e^y = 0. For any values of y, the exponential function cannot assume 0

Thus, the function f do not have maxima or minima.

Que.5. For the given functions find the absolute maximum and minimum values in the given intervals.

(i)f(y) = y^3, y ϵ (-2,2)

f'(y) = $3y^3$

f'(y) = 0 ⇒ y = 0

f(0) = 0

f(-2) = $(-2)^3$ = -8

f(2) = $(2)^3$ = 8

Thus, the absolute maximum is 8 and the absolute minimum is -8 for the given interval.

(ii) f(y) = $(y – 1)^2$ + 3, y ϵ (-3,1)

f'(y) = 2(y – 1)

f'(y) = 0

2(y – 1) = 0

y = 1

The given intervals are (-3,1)

f(1) = $(1 – 1)^2$ + 3 = 0 + 3 = 3

f(-3) = $(–3 – 1)^2$+ 3 = 16 + 3 = 19

Thus, the absolute maximum is 19 and the absolute minimum is -3 for the given interval.

Que.6. The profit function is p(y) = 41 – 34y – 18y^2. Find the maximum profit.

Ans. p' (y) = -24 – 36y

f''(y) = -36

p'(y) = 0

y = $\frac{-24}{36}$ = $\frac{-2}{3}$

f''($\frac{-2}{3}$) = -36 < 0

By second derivative

The local maxima point is y = $\frac{-2}{3}$

Maximum profit = p(= $\frac{-2}{3}$)

= 41 – 24($\frac{-2}{3}$

) – 18 $(\frac{-2}{3})^2$

= 41 + 16 – 8

= 49

The maximum profit is 49 units.

Que.7. The maximum value for the function sin 2y can be attained at what point in the interval (0,2n).

Ans. f(y) = sin 2y

f'(y) = 2cos 2y

f'(y) = 0

cos 2y = 0

2y = $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, $\frac{7\pi}{2}$

y = $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$

f($\frac{\pi}{4}$) = sin $\frac{\pi}{2}$ = 1

f($\frac{3\pi}{4}$) = sin $\frac{3\pi}{2}$ = -1

f($\frac{5\pi}{4}$) = sin $\frac{5\pi}{2}$ = 1

f($\frac{7\pi}{2}$) = sin $\frac{7\pi}{2}$ = -1

f(0) = sin 0 = 0, f(2π) = sin 2π = 0

Thus, the absolute maximum is occurring at y = $\frac{\pi}{4}$ and y = ($\frac{5\pi}{4}$).

Que.8. In a function sin y + cos y, find the maximum value.

f(y) = sin y + cos y

Ans. f'(y) = cos y – sin y

f'(y) = 0

sin y = cos y = – (sin y + cos y) $\Rightarrow$ tan y = 1

y = $\frac{\pi}{4}$, $\frac{5\pi}{4}$

f''(y) = – sin y – cos y = – (sin y + cos y)

When (sin y + cos y) is positive, f''(y) will be negative.

f''(y) is negative when y ϵ (0, $\frac{\pi}{2}$)

Thus, y = $\frac{\pi}{4}$

$f''(\frac{\pi}{4}) = -(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}) = -(\frac{2}{\sqrt{2}}) = -\sqrt{2} < 0$

By second derivative, maximum is at y = $\frac{\pi}{4}$

$f(\frac{\pi}{4}) = \sin\frac{\pi}{4}$

$+ \cos\frac{\pi}{4} = \sqrt{2}$

Que.9. In the interval (1, 3) and (-3, -1), find the maximum value of $2y^3 - 24y + 107$.

Ans. $f(y) = 2y^3 - 24y + 107$

$f'(y) = 6y^2 - 24 = 6(y^2 - 4)$

$f'(y) = 0$

$6(y^2 - 4) = 0$

$y = \pm 2$

If we consider the interval (1, 3)

f(2) = 2(8) – 24(2) + 107 = 16 – 48 + 107 = 75

f(1) = 2(1) – 24(1) + 107 = 2 – 24 + 107 = 85

f(3) = 2(27) – 24(3) + 107 = 54 – 72 + 107 = 89

Therefore, the absolute maximum value of f(y) in interval (1, 3) is 89 at y = 3

Now, if we consider the interval (-3, -1)

f(-3) = 2 (-27) – 24 (-3) + 107 = -54 + 72 + 107 = 125

f(-1) = 2 (-1) – 24 (-1) + 107 = -2 + 24 + 107 = 129

f(-2) = 2 (-8) – 24 (-2) + 107 = -16 + 28 + 107 = 139

Therefore, in the interval (-3, -1), the maximum value is 139 at y = -2.

Que.10. For the function $y^4 - 62y^2 + ay + 9$, the maximum value is attained at y = 1 in the interval (0, 2). Find the value of a.

Ans. $f(y) = y^4 - 62y^2 + ay + 9$

$f'(y) = 4y^3 - 124y + a$

Maximum value is attained at y = 1

$f'(y) = 0$

4 – 124 + a = 0

a = 120

Therefore, 120 is the maximum value

Que.11. For the function y + sin 2y, find the maximum and minimum value at the interval (0, 2n).

Ans. f(y) = y + sin 2y

$f'(y) = 1 + 2\cos 2y$

$f'(y) = 0$

$\cos 2y = -\frac{1}{2} = -\cos\frac{\pi}{2} = \cos(\pi - \frac{\pi}{3}) = \cos\frac{2\pi}{3}$

$2y = 2n \pm \frac{2\pi}{3}$, n ϵ Z

$y = n\pi \pm \frac{\pi}{3}$, n ϵ Z

$y = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \in (0, 2\pi)$

$f(\frac{\pi}{3}) = \frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

$f(\frac{2\pi}{3}) = \frac{2\pi}{3} + \sin\frac{4\pi}{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

$f(\frac{4\pi}{3}) = \frac{4\pi}{3} + \sin\frac{8\pi}{3} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$

$f(\frac{5\pi}{3}) = \frac{5\pi}{3} + \sin\frac{10\pi}{3} =$

$\frac{5\pi}{3} - \frac{\sqrt{3}}{2}$

f(0) = 0 + sin 0 = 0

$f(2\pi) = 2\pi + \sin 4\pi = 2\pi + 0 = 2\pi$

Therefore, the absolute maximum and minimum value is 2n and 0 in the interval (0, 2n)

Que.12. A sum of two numbers in 24 whose product is as large as possible. Find the two numbers.

Ans. Let one number be y. Then, the other number is (24 – y)

P(y) is the product of two numbers.

$P(y) = y(24 – y) = 24y – y^2$

$P'(y) = 24 – 2y$

$P''(y) = -2$

$P'(y) = 0$

$y = 12$

$P''(12) = -2 < 0$

By second derivative, the local maxima of P is at point y = 12. Thus, the product of two numbers is maximum when there are 12 and 24.

Que.13. If y + z = 60 and yz^3 is maximum, find the two positive numbers.

Ans. $y + z = 60$

$y = 60 – y$

$f(y) = yz^3$

$f(y) = x(60 – y)^3$

$f'(y) = (60 – y)^3 – 3y(60 – y)^2$

$= (60 – y)^2 (60 – y – 3y)$

$= (60 – y)^2 (60 – 4y)$

$f''(y) = -2(60 – y) (60 – 4y) – 4(60 – y)^2$

$= -2 (60 – y) (60 – 4y + 2(60 – y))$

When y = 60, $f''(y) = 0$.

When y = 15, $f''(y) = -12(60 – 15) (30 – 15) = -12 \times 45 \times 15 < 0$

$f'(y) = 0$

y = 60 or y = 15

By second derivative, the local maxima is at the point y = 15.

Thus the required numbers are y =15 and z = 60 – 15 = 45.

Que.14. The sum of two positive numbers is 16 and whose cube is minimum. Find the two numbers.

Ans. Let y be a number. The other number is (16 – y)

S(y) is the sum of the cubes.

$S(y) = y^3 + (16 – y)^3$

$s'(y) = 3y^2 – 3(16 – y)^2 = 0$

$y^2 – (165 – y)^2 = 0$

$y^2 – 256 – y^2 + 32y = 0$

$y = \frac{256}{32} = 8$

$S''(8) = 6(8) + 6(16 – 8) = 48 + 48 = 96 > 0$

By second derivative, the local minima is at point y = 8.

Thus, the numbers are 8 and 16 − 8 = 8.

Que.15. A box is made without the top from a square piece of tin whose sides are 18 cm by cutting a square from each corner and folding up the flaps. The volume of box should be maximum, so what length of sides are to be cut?

Ans. Let y be the length to be cut. So, the length and breadth are (18 – 2y) cm and the height is y cm.

$V(y) = y(18 – 2y)^2$

$V'(y) = (18 – 2y)^2 – 4y(18 – 2y)$

= (18 – 2y) (18 – 6y)

= 12(9 – y) (3 – y)

V''(y) = 12(– (9 – y) – (3 – y))

= –12(9 – y + 3 – y)

= – 12(12 – 2y)

= – 24(6 – y)

V'(y) = 0

y = 9 or y = 3

If y = 9, then the length and breadth will become 0

y $\neq$ 9

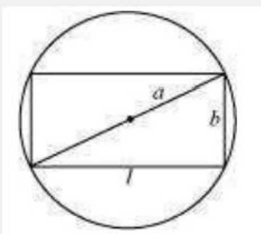
y = 3

V''(y) = -24 (6 – 3) = –72 < 0

By second derivative, the point of maxima is y = 3

Thus, 3cm should be removed from the sides.

Que.16. Prove that the all the rectangles inscribed inside a circle, square has the maximum area



Ans. Let the radius be a.

The diagonal passes through the center whose length is 2a cm.

By Pythagoras theorem,

$(2a)^2 = l^2 + b^2$

$b^2 = 4a^2 – l^2$

b = $\sqrt{4a^2 - l^2}$

Area A = l$\sqrt{4a^2 - l^2}$

$\frac{dA}{dl} = \sqrt{4a^2 - l^2} + l\,\frac{1}{2\sqrt{4a^2-l^2}}\,(-2l) = \sqrt{4a^2 - l^2} - \frac{l^2}{\sqrt{4a^2-l^2}}$

$= \frac{4a^2-2l^2}{\sqrt{4a^2-l^2}}$

$\frac{d^2A}{dl^2} = \frac{\sqrt{4a^2-l^2}(-4l)-(4a^2-2l^2)\frac{-2l}{2\sqrt{4a^2-l^2}}}{4a^2-l^2}$

$= \frac{(4a^2-l^2)(-4l)+(4a^2-2l^2)}{(4a^2-l^2)^{\frac{3}{2}}}$

$= \frac{-2l(6a^2-l^2)}{(4a^2-l^2)^{\frac{3}{2}}}$

Now, $\frac{dA}{dl} = 0$

$4a^2 = 2l^2$

l = $\sqrt{2}a$

b = $\sqrt{4a^2 - 2a^2} = \sqrt{2}a$

When l = $\sqrt{2}a$

$\frac{d^2A}{dl^2} = \frac{-2(\sqrt{2}a)(6a^2-2a^2)}{2\sqrt{2}a^3} = \frac{-8\sqrt{2}a^3}{2\sqrt{2}a^3} = -4 < 0$

By second derivative, when l = $\sqrt{2}a$

Area of the rectangle is maximum.

Since l = b = $\sqrt{2}a$, the rectangle is a square.

Hence, it has been proved that of all the rectangles inscribed in the given fixed circle, the square has the maximum area.

Que.17. Show that the height of the right circular cylinder of given surface and maximum volume is equal to the diameter of the base.

Ans. Let a and h be the radius and height of the cylinder.

S = $2\pi a^2 + 2\pi ah$

h = $\frac{S-2\pi a^2}{2\pi a}$

= $\frac{S}{2\pi}\frac{1}{a} - a$

V = $\pi a^2 h = \pi a^2(\frac{S}{2\pi}\frac{1}{a} - a) = \frac{Sa}{2} - \pi a^3$

$\frac{dV}{da} = \frac{S}{2} - 3\pi a^2$

$\frac{d^2V}{da^2} = -6\pi a$

$\frac{dV}{da} = 0$

$\frac{S}{2} = 3\pi a^2$

$a^2 = \frac{S}{6\pi}$

When, $a^2 = \frac{S}{6\pi}$, then $\frac{d^2V}{da^2} = -6\pi\sqrt{\frac{S}{6\pi}} < 0$

By second derivative, when $a^3 = \frac{S}{6\pi}$ it has the maximum value.

When $a^2 = \frac{S}{6\pi}$, then h = $\frac{6\pi a^2}{2\pi}\frac{1}{a} - a = 2a$.

Thus, when the height is twice the radius, the volume is maximum.

Que.18. For all right circular closed cylinder of volume 100 cubic cm, find the dimension of can which has the minimum surface area.

Ans. Let a and h be the radius and height

V = $\pi a^2 h = 100$

h = $\frac{100}{\pi a^2}$

S = $\pi a^2 h + 2\pi ah = 2\pi a^2 + \frac{200}{a}$

$\frac{dS}{da} = 4\pi a + \frac{200}{a^2}$

$\frac{d^2S}{da^2} = 4\pi + \frac{400}{a^3}$

$\frac{dS}{da} = 0$

$4\pi a = \frac{200}{a^2}$

$a^3 = \frac{200}{4\pi} = \frac{50}{\pi}$

$a = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$

Since $a = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$, $\frac{d^2S}{da^2} > 0$

By second derivative, When the radius is $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm, the surface area is minimum.

When $a = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$, $h = \frac{100}{\pi\left(\frac{50}{\pi}\right)^{\frac{2}{3}}} = 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

Therefore, the dimensions are $a = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm

$h = 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm

Que.19. A wire is cut into two pieces whose length is 28 m. A circle and a square are made from those pieces. What length should be the pieces such that the combined area of the circle and the square is minimum?

Ans. Let l be the length of one piece.

So, the length of the other piece is (28 – l) m

Side of square = $\frac{l}{4}$

Radius of the circle is a

$2\pi a = 28 - l$

$a = \frac{1}{2\pi}(28 - l)$

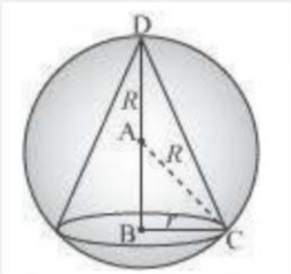
Total area A = $\frac{l^2}{16} + \pi\left(\frac{1}{2\pi}(28 - l)\right)^2$

$=\frac{l^2}{16}+\frac{1}{4\pi}(28-l)^2$

$\frac{dA}{dl}=\frac{2l}{16}+\frac{2}{4\pi}(28-l)(-1)=\frac{l}{8}-\frac{1}{2\pi}(28-l)$

$\frac{d^2A}{dl^2}=\frac{1}{8}+\frac{1}{2\pi}>0$

Que.20. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of thevolume of the sphere.



Ans:

Let r and h be the radius and height of the cone respectively inscribed in a sphere of radius R.

Let V be the volume of the cone.

Then, $\frac{1}{3}\Pi r^2 h$

Height of the cone is given by,

h = R + AB = R + $\sqrt{R^2-r^2}$

$V=\frac{1}{3}\Pi r^2\left(R+\sqrt{R^2-r^2}\right)$

$=\frac{1}{3}\Pi r^2 R+\frac{1}{3}\Pi r^2\sqrt{R^2-r^2}$

$\frac{dV}{dr}=\frac{2}{3}\Pi rR+\frac{2}{3}\Pi r\sqrt{R^2-r^2}+\frac{1}{3}\Pi r^2.\frac{-2r}{2\sqrt{R^2-r^2}}=\frac{2}{3}\Pi rR+\frac{2}{3}\Pi r\sqrt{R^2-r^2}-\frac{1}{3}\Pi\frac{r^3}{R^2-r^2}=\frac{2}{3}\Pi rR+\frac{2\Pi r(R^2-r^2)-\Pi r^3}{3\sqrt{R^2-r^2}}=\frac{2}{3}\Pi rR+\frac{2\Pi rR^2-3\Pi r^3}{3\sqrt{R^2-r^2}}$

$\frac{d^2V}{dr^2}=\frac{2\Pi R}{3}+\frac{3\sqrt{R^2-r^2}(2\Pi R^2-9\Pi r^2)-(2\Pi rR^2-3\Pi r^3).\frac{(-2r)}{6\sqrt{R^2-r^2}}}{9(R^2-r^2)}\Rightarrow 2R=\frac{3r^2-2R^2}{\sqrt{R^2-r^2}}\Rightarrow 2R\sqrt{R^2-r^2}=3r^2-2R^2\Rightarrow 4R^2\left(R^2-r^2\right)=(3r^2-2R^2)^2$
$\Rightarrow 4R^4-4R^2r^2=9r^4+4R^4-12r^2R^2\Rightarrow 9r^4=8R^2r^2\Rightarrow r^2=\frac{8}{9}R^2$

When $r^2=\frac{8}{9}R^2$, then $\frac{d^2V}{dr^2}$<0.

by second derivative test, the volume of thecone is the maximum when $\Rightarrow r^2=\frac{8}{9}R^2$.

$when r^2=\frac{8}{9}R^2, h=R+\sqrt{R^2-\frac{8}{9}R^2}=R+\sqrt{\frac{1}{9}R^2}=R+\frac{R}{3}=\frac{4}{3}R$

Therefore,

= $\frac{1}{3}\Pi\left(\frac{8}{9}R^2\right)\left(\frac{4}{3}R\right)$

= $\frac{8}{27}\left(\frac{4}{3}\Pi R^3\right)$

=$\frac{8}{27}$ x (volume of the sphere)

Hence, the volume of the largest cone that can be inscribed in a sphere is $\frac{8}{27}$ the volume of the sphere.

Que.21. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

Ans:

Let r and h be the radius and the height(altitude) of the cone respectively. Then, the volume(V) of the cone is given as:

V=$\frac{1}{3\prod}\prod r^2h\Rightarrow h=\frac{3V}{v^2}$

The surface area(S) of the cone is given by,

S=nrl (where l is the slant height)

$=\Pi r\sqrt{r^2+h^2}$

$=\Pi r\sqrt{r^2+\frac{9V^2}{\Pi^2r^4}}=\Pi\frac{r\sqrt{9^2r^6+V^2}}{\Pi r^2}$

$\frac{1}{r}\sqrt{\Pi 2r^6+9V^2}\ \frac{ds}{dr}=\frac{\frac{r}{.}6\Pi^2r^5 2\Pi\sqrt{r^6 9V^2}-\sqrt{\Pi 2}r^6+9V^2}{r^2}=\frac{3\Pi^2r^6-\Pi^2r^6-9V^2}{r^2\sqrt{\Pi^2r^6+9V^2}}$

$==\frac{2\Pi^2r^6-9V^2}{r^2\sqrt{\Pi^2r^6+9V^2}}$

Now, $\frac{ds}{dr}=0\Rightarrow 2\Pi^{\cdot}2r^6=9V^2\Rightarrow r^6=\frac{9V^2}{2\Pi^2}$

Thus, it can be easily verified that when $r^6=\frac{9V^2}{\ }\ \frac{d^2S}{\ }>0$

Thus, it can be easily verified that when ... $2\Pi^2$, dr^2 > 0.

By second derivative test, the surface area of the cone is the least when

$\frac{3V}{\Pi r^2} = \frac{3}{\Pi r^2}(\frac{2\Pi^2 r^6}{9})^{\frac{1}{2}} = \frac{3}{\Pi r^2}.\frac{\sqrt{2\Pi r^3}}{3} = \sqrt{2r}$

Hence, for a given volume, the right circular cone of the least curved surface has an altitude equal to $\sqrt{2}$ time the radius of the base.

$\frac{dA}{dl}$ = 0

(π + 4)l – 112 = 0

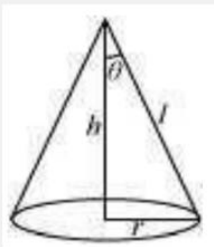
l = $\frac{112}{\pi+4}$

Thus, when l = $\frac{112}{\pi+4}$, $\frac{d^2A}{dl^2}$ > 0.

By second derivative when l = $\frac{112}{\pi+4}$, the area is minimum.

The length of the wire used in making the circle is 28 – $\frac{112}{\pi+4}$ = $\frac{28\pi}{\pi+4}$ cm.

Que.22. Show that the semi vertical angle of the cone of the maximum volume and of given slant height is $tan^{-1}\sqrt{2}$.



Ans: Let Θ be the semi-vertical angle of the cone.

It is clear that Θ $\in \left[0, \frac{\Pi}{2}\right]$.

Let r, h and l be the radius height and the slant height of the cone respectively.

The slant height of the cone is given as constant.

Now, r = l sin Θ and h = l cos Θ.

The volume(v) of the cone is given by,

$v = \frac{1}{3}\prod r^2 h$

= $\frac{1}{3}\prod(l^2 sin^2\theta)(lcos\theta)$

= $\frac{1}{3}\prod l^3 sin^2\theta cos\theta$

$\frac{dv}{d\theta} = \frac{l^3\prod}{3}\left[sin^2\theta(-sin\theta) + cos\theta(2sin\theta cos\theta)\right]$

=$\frac{l^3\prod}{3}\left[-sin^3 + 2sin\theta cos^2\theta\right]$

$\frac{d^2v}{d\theta^2} = \frac{l^3\prod}{3}\left[-3sin^2\theta cos\theta + 2cos^3\theta - 4sin^2\theta cos\theta\right]$

= = $\frac{l^3\prod}{3}\left[2cos^3\theta - 7sin^2\theta cos\theta\right]$

Now, $\frac{dv}{d\theta} = 0$

$\Rightarrow sin^3\theta = 2sin\theta cos^2\theta \Rightarrow tan^2\theta = 2 \Rightarrow tan\theta = \sqrt{2} \Rightarrow \theta = tan^{-1}\sqrt{2}$

Now, when $\theta = tan^{-1}\sqrt{2}$, then $\Rightarrow tan^2\theta = 2$

Or $\Rightarrow sin^2\theta = 2cos^2\theta$.

Then we have:

$\frac{dv^2}{d\theta^2} = 2ywhichisnearesttothepoint = \frac{l^3\prod}{3}\left[2cos^3\theta - 14cos^3\theta\right] = -4\prod l^3 cos^3\theta < 0 for\theta\varepsilon[0, \frac{\Pi}{2}]$

By second derivative test, the volume(v) is the maximum when $\Rightarrow \theta = tan^{-1}\sqrt{2}$.

Hence, for a given slant height, the semi- vertical angle of the cone of the maximum volume is $tan^{-1}\sqrt{2}$.

Que.23. The point on the curve x^2= 2y ehich is nearest to the point (0 , 5) is

(A) $(2\sqrt{2}, 4)$

(B) $(2\sqrt{2}, 0)$

(C) (0,0)

(D) (2,2)

Ans:

The given curve is $x^2= 2y$.

For each value of x, the position of the point will be $(x, \frac{x^2}{2})$.

The distance d(x) between the points $(x, \frac{x^2}{2})$. and (0,5) is given by,

$$d(x) = \sqrt{(x-0)^2 + (\frac{x^2}{2}-5)^2)} = \sqrt{x^2 + \frac{x^4}{4} + 25 - 5x^2} = \sqrt{\frac{x^4}{4} - 4x^2 + 25}.$$

$$d'(x) = \frac{(x^3-8x)}{2\sqrt{\frac{x^4}{4}-4x^2+25}} = \frac{(x^3-8x)}{\sqrt{x^4-16x^2+100}}$$

Now, d''(x) = 0$\Rightarrow$ x^3-8x=0

$\Rightarrow$ x(x^2-8)=0

$\Rightarrow$ x=0, $\pm 2\sqrt{2}$.

$$And, d''(x) = \frac{\sqrt{x^4-16x^2+100}(3x^2-8)-(x^3-8x).\frac{4x^3-32x}{2\sqrt{x^4-16x^2+100}}}{(x^4-16x^2+100)}$$

$$= \frac{(x^4-16x^2+100)(3x^2-8)-2(x^3-8x)(x^3-8x)}{(x^4-16x^2+100)^{\frac{3}{2}}}$$

$$= \frac{(x^4-16x^2+100)(3x^2-8)-2(x^3-8x)^2}{(x^4-16x^2+100)^{\frac{3}{2}}}$$

When, x = 0, then d''(x)= $\frac{(36(-8))}{6^3} < 0$

When, x = $\pm$ 2 $\sqrt{2}$,d''(X)>0.

by second derivative test,d(x) is the minimum at x = $\pm 2\sqrt{2}$

When x=$\pm 2\sqrt{2}$,y=$\frac{(2\sqrt{2})^2}{2} = 4$.

Hence, the point on the curve x^2=2y which is nearest to the point(0,5) is ($^+_-$2 $\sqrt{2}$,4).

The correct answer is A.

Que.24. For all real values of x, the minimum value of $\frac{1-x+x^2}{1+x+x^2} is$

(A) 0

(B) 1

(C) 3

(D) $\frac{1}{3}$

Ans:

Let f(x)= $\frac{1-x+x^2}{1+x+x^2} is$

$$f'(x) = \frac{(1+x+x^2)(-1+2x)-(1-x+x^2)(1+2x)}{(1+x+x^2)^2}$$

$$= \frac{-1+2x-x+2x^2-x^2+2x^3-1-2x+x+2x^2-x^2-2x^3}{(1+x+x^2)^2}$$

$$= = \frac{2x^2-2}{(1+x+x^2)^2} = \frac{2(x^2-1)}{(1+x+x^2)^2}$$

$$f'(x) = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

Now, f''(x) = $\frac{2\left[(1+x+x^2)^2(2x)-(x^2-1)(2)(1+x+x^2)(1+2x)\right]}{(1+x+x^2)^4}$

$$= \frac{4(1+x+x^2)[(1+x+x^2)x-(x^2-1)(1+2x)]}{(1+x+x^2)^4} = \frac{4[x+x^2+x^3-x^2-2x^3+1+2x]}{(1+x+x^2)^3} = \frac{4[1+3x-x^3]}{(1+x+x^2)^3}$$

And, f''(1)= $\frac{4(1+3-1)}{(1+1+1)^3} = \frac{4(3)}{(3)^3} = \frac{4}{9} > 0$

Also, f''(-1)= $\frac{4(1-3+1)}{(1-1+1)^3} = 4(-1) = -4 < 0$

by second derivative test, f is the minimum at x = 1 and the minimum value is given by f(1)= $\frac{1-1+1}{1+1+1}$= $\frac{1}{3}$

The correct ans is D.

Que.25. The maximum value of $[x(x-1)+1]^{\frac{1}{3}}, 0 \leq x \leq 1$ is

(A) $\frac{1}{1}3)^{\frac{}{3}}$

(B) $\frac{1}{2}$

(C) 1

(D) 0

Ans:

Let f(x)= $[x(x-1)+1]^{\frac{1}{3}}$

$f'(x)=\left[\frac{2x-1}{3[x(x-1)+1]}\right]^{\frac{2}{3}}$

Now, f'(x)=0⇒ $x=\frac{1}{2}$

Then, we evaluate the value of f at critical point $x=\frac{1}{2}$ and at the end points of the interval [0,1] {i.e. at x=0 and x=1).

$f(0)=[0(0-1)+1]^{\frac{1}{3}}=1\ f(1)=[1(1-1)+1]^{\frac{1}{3}}=1\ f(\frac{1}{2})=\left[\frac{1}{2}(\frac{-1}{2})+1\right]^{\overline{3}}=(\frac{3}{4})^{\frac{1}{3}}$

Hence, we can conclude that the maximum value of f in the interval[0,1] is 1.

The correct answer is C.